

STRATA HASSE INVARIANTS, HECKE ALGEBRAS AND GALOIS REPRESENTATIONS

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ABSTRACT. We define a stack $G\text{-ZipFlag}^\mu$ above the Pink-Wedhorn-Ziegler stack of G -zips, fibered in flag varieties, which is a group-theoretical generalization of the flag space introduced in [EvdG09] for Siegel-type Shimura varieties. For each Ekedahl-Oort stratum of a general Hodge-type Shimura variety, we construct a Hecke-equivariant section of the Hodge line bundle which (set-theoretically) cuts out the smaller strata in its closure. We call these sections “group-theoretical Hasse invariants”. Using them as our main tool, we are able to: (1) Attach Galois representations to many automorphic representations with non-degenerate limit of discrete series archimedean component, generalizing ([DS74], [Tay91], [Jar97], [Gol14],[GN14]) (2) Attach pseudo-representations to torsion classes in the coherent cohomology of many Hodge-type Shimura varieties, generalizing [ERX] (3) Prove that all Ekedahl-Oort strata are affine, both for a general compact Shimura variety of Hodge-type and for the minimal compactification of a Siegel-type Shimura variety, thereby proving a conjecture of Oort. (4) Generalize (part of) Serre’s letter to Tate on mod p forms [Ser96] to general Hodge-type Shimura varieties, refining (parts of) previous results in some PEL cases ([Ghi04], [Red03]).

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1. INTRODUCTION

In this work, we construct group-theoretical Hasse invariants, which constitute a vast, multi-directional generalization of the classical Hasse invariant of modular curves. Our motivation originated from two seemingly distant themes: (i) The facet of the Langlands Program concerned with attaching Galois representations to (algebraic) automorphic representations, and (ii) the geometric study of the Ekedahl-Oort stratification and related objects stemming from the special fiber of a (Hodge-type) Shimura variety. Prior to this paper, much of the literature has kept (i) and (ii) separate. During the course of this work, we discovered that group-theoretical Hasse invariants reveal a profound link between the two themes (i) and (ii). By exploiting this link, we are able to prove prominent conjectures in each of (i) and (ii). It may very well be that our results are only the tip of the iceberg, and that following the link suggested by Hasse invariants to its source will lead to further, deeper results.

We took two paths, one stemming from each theme, which led us to the group theoretical Hasse invariants of this paper. Concerning the *irregular case* of (i), in the work of Deligne-Serre [DS74] and its descendants, the most crucial property of the Hasse invariant and its generalizations is that these mod p automorphic forms are *Hecke-equivariant*. On the other hand, the recent construction [GN14], and in greater generality [KW14], of generalized Hasse invariants whose non-vanishing locus is precisely the μ -ordinary locus suggested to us that generalized Hasse invariants shed light on (ii) via their non-vanishing locus. At this point, we were curious to find sections of closed subschemes of the special fiber of a Shimura variety which were *either* (a) Hecke equivariant *or* (b) whose non-vanishing loci were directly related to the Ekedahl-Oort stratification. To our surprise, any section that we found satisfying one of these two properties also satisfied the other. Moreover, all of the sections that we found turned out to be of group-theoretic origin, even in cases where we first came across the sections by a different approach. These discoveries have led us to question whether *all* sections which satisfy one of (a), (b) are necessarily of group-theoretic origin, which would imply that they also satisfy the second. As we conclude writing this paper, this question remains open, but we hope to return to it in future work.

We shall now briefly review the history of our two topics (§§1.1-1.2). We then comment on how this work arose, and on its relationship with other preprints and announcements (§1.3). Finally, the structure of this paper is outlined in §1.4.

1.1. The association of Galois representations to automorphic representations. The Langlands Program has played an increasingly fundamental role across mathematics for over fifty years. Among its many facets, the association of Galois representations to (algebraic) automorphic representations over number fields has been a driving force. One may argue that the ultimate goal of the Langlands correspondence is a correspondence with (pure) motives, rather than Galois representations. However, to this day the only known way to state that a motive corresponds to an automorphic representation is via the Galois representations afforded by its ℓ -adic realizations. Furthermore, the association of a Galois representation has proven to be a much more tractable problem than that of associating a motive. The main reason for this is that to merely construct a Galois representation one

may appeal to congruence (and ℓ -adic limit) arguments, while no one knows if such techniques are applicable for constructing motives.

Beyond the case of $GL(1)$, the construction of automorphic Galois representations has as its historical foundation the works of Deligne [Del69] and Deligne-Serre [DS74] on classical modular forms of weight at least two and weight one, respectively. To this day, all of the generalizations to other automorphic representations have followed, at least to some extent, the blueprint laid out by Deligne and Deligne-Serre.

1.1.1. The archimedean component hierarchy. Let \mathbf{G} be a connected, reductive \mathbf{Q} -group and let π be an automorphic representation with archimedean component π_∞ . Let us assume that π is C -algebraic in the sense of [BG11]. There has been ample evidence that the representation-theoretic nature of π_∞ is the main indicator of the difficulty of attaching a Galois representation to π . Already in the case of $GL(2)$, the case of weight one proved to be considerably more intricate than the case of weight at least two. Moreover, the case of Maass forms of eigenvalue $1/4$, which also pertains to $GL(2)$ remains open to this day. In terms of π_∞ , the three cases for $GL(2)$, (i) weight at least two, (ii) weight one, (iii) Maass eigenvalue $1/4$ are characterized as (i) discrete series, (ii) holomorphic limit of discrete series, (iii) not a limit of discrete series.

For more general groups, the complexity of the archimedean component is measured by several invariants. The coarsest, yet most determinative, invariant of the archimedean component is whether its infinitesimal character is regular or not. By far the most progress in attaching Galois representations has been achieved when π is regular. By a long series of works, much is now known in the regular case. For the best known results in the regular case, see [Shi11], [Art13], [HLTT13] and [Sch].

On the other hand, prior to our work and to that of Pilloni-Stroh [PS] (see §1.3.2), quite little was known in the irregular case. The work of Schmid, Williams, Knapp-Zuckerman, Mirkovic and Harris (see §2.2) showed that the part of the irregular case where π_∞ is a non-degenerate limit of discrete series is the most tractable, because in that case certain technical assumptions imply a relationship with the coherent cohomology of Shimura varieties. As of this writing, no irregular case has been treated beyond non-degenerate limits of discrete series.

Non-degenerate limits of discrete series can be further separated into holomorphic and non-holomorphic. In terms of the coherent cohomology of Shimura varieties, holomorphic (resp. non-holomorphic) translates into cohomology in degree zero (resp. strictly positive degree). The aforementioned work of Deligne-Serre on weight one forms falls into the category of a holomorphic limit of discrete series. Using the critical notion of pseudo-representations due to Wiles [Wil88a] and Taylor [Tay91], the Deligne-Serre method was generalized to holomorphic limits for the group $GSp(4)$ by Taylor in *loc. cit.*, and for $GL(2)$ over a totally real field by Jarvis [Jar97]. Still in the holomorphic case, these works were subsequently generalized by the first author and jointly with Marc-Hubert Nicole to general unitary groups [Gol14], [GN14].

All of the works above on the holomorphic case build on the original strategy of Deligne-Serre of constructing congruences between the irregular π and many regular π' . In this way, one tries to reduce the desired result to a corresponding one for π' . Moreover, in all of these results, the congruences are constructed by multiplying by a mod p automorphic form which is Hecke-equivariant. This mod p

automorphic form is constructed as a section of a power of the Hodge line bundle on the special fiber of a Shimura variety associated to G . It is a generalization of the Hasse invariant, see §1.2.

The non-holomorphic case reveals genuinely new obstacles. Due to the degree of cohomology being non-zero, it becomes apparent that multiplication by a single mod p automorphic form is insufficient. Our new idea to overcome this problem is to use not just one mod p automorphic form, but rather a whole family of such. Moreover, these mod p automorphic forms are not defined on the whole special fiber of the Shimura variety, but rather on the closures of Ekedahl-Oort strata of the special fiber. We are then able to reduce to the holomorphic case by using an inductive procedure, where exact sequences in cohomology are used to produce relations among the cohomologies of these Hecke-stable subschemes.

In short, under certain technical hypotheses, we prove a reduction theorem, which in terms of π_∞ reduces from the non-degenerate, non-holomorphic case to the holomorphic one. In terms of cohomology, this corresponds to the reduction from arbitrary degree to degree zero.

1.1.2. *Torsion.* A problem which is closely related to that discussed in §1.1.1 is to attach Galois pseudo-representations to the coherent cohomology of a Shimura variety modulo a prime power. This is in some sense a more sophisticated version of attaching Galois representations to automorphic ones. Already in the initial example of classical modular forms, it was observed that when these are defined using integral models of modular curves, there are mod p forms of weight one that are not the reduction modulo p of a characteristic zero weight one form. Nevertheless, the work of Carayol [Car94] showed that such non-liftable forms modulo a prime power still admit an associated pseudo-representation¹.

Since then, there have been a number of results and conjectures about the existence of pseudo-representations for (i) the coherent cohomology of Shimura varieties modulo a prime power and (ii) the Betti cohomology of a locally symmetric manifold (*cf.* the introduction to [ERX]). In *loc. cit.*, the coherent cohomology of Hilbert-type Shimura varieties was treated. We note in connection with §1.1.1, that the work [ERX] did not treat any new automorphic representations, because in the Hilbert case, π_∞ may always be arranged to be \mathbf{X} -holomorphic with a suitable choice of \mathbf{X} , in the terminology of [Gol14]. The work [ERX] also used several mod p , Hecke-equivariant automorphic forms, but in a different way than here, that does not seem to generalize beyond the Hilbert case.

Using totally different methods, namely his theory of perfectoid spaces, Scholze [Sch] treated the case of the Betti cohomology of a locally symmetric manifold associated to $GL(m)$.

In this work, we show that the Hecke algebra acting on the coherent cohomology of a Hodge-type Shimura variety modulo a prime power receives a pseudo-representation, under the same technical hypotheses that we need for attaching Galois representations to automorphic representations.

1.2. The Ekedahl-Oort stratification and Hasse invariants.

¹Strictly speaking, Carayol did not use the language of pseudo-representations and did not consider weight one forms, but his results trivially adapt to this context.

1.2.1. *Shimura varieties.* Shimura varieties have played a key role in the Langlands program, as objects whose cohomology realizes instances of the Langlands correspondence. They were introduced by Goro Shimura in the 1960's. Initially defined as complex manifolds, they can be endowed with a natural structure of a complex algebraic variety by a theorem of Baily-Borel. Furthermore, they admit a model over a number field (the reflex field), which yields an action of a Galois group on these objects. In the definition given by Deligne, a Shimura variety is attached to a Shimura datum (\mathbf{G}, \mathbf{X}) where \mathbf{G} is a connected reductive group over \mathbf{Q} and \mathbf{X} is a $\mathbf{G}(\mathbf{R})$ -conjugacy class of homomorphisms $\mathbf{S} \rightarrow \mathbf{G}_{\mathbf{R}}$ (where $\mathbf{S} = \text{Res}_{\mathbf{C}/\mathbf{R}}(\mathbf{G}_{m, \mathbf{C}})$), satisfying a list of axioms (see [Del77] §1.5). The attached Shimura variety is then a tower of varieties $Sh(\mathbf{G}, \mathbf{X}) = (Sh_{\mathcal{K}}(\mathbf{G}, \mathbf{X}))_{\mathcal{K}}$ for \mathcal{K} a compact open subgroup of $\mathbf{G}(\mathbf{A}_f)$. One obtains an action of $\mathbf{G}(\mathbf{A}_f)$ on various objects attached to the Shimura variety $Sh(\mathbf{G}, \mathbf{X})$. Hence Shimura varieties combine a Galois action as well as an action of the group $\mathbf{G}(\mathbf{A}_f)$ on the automorphic side.

1.2.2. *The Ekedahl-Oort stratification.* The construction of integral models of Shimura varieties was studied in detail by Kottwitz in [Kot92], where he introduces the general framework of PEL-type Shimura varieties and constructs models at primes of good reduction. This construction was then generalized by Kisin-Vasiu to the Hodge-type case in [Kis10] and [Vas99]. The special fiber of the Shimura variety is then a smooth quasi-projective scheme over a finite field. For Siegel-type Shimura varieties, Oort defined in [Oor01] a stratification of the special fiber by isomorphism classes of the p -torsion of the p -divisible group (the so-called truncated Barsotti-Tate group of level 1, in short BT1). In his paper, the strata are parametrized by "elementary sequences". Later on, Moonen gave in [Moo01] a group-theoretical classification of BT1's with PEL-structure (of type μ), which describes the isomorphism classes as a certain subset ${}^I W \subset W$ of the Weyl group of the automorphism group of the PEL-structure.

In a series of papers, Moonen, Wedhorn, Pink and Ziegler showed that BT1's with G -structure and type μ give rise to an algebraic stack $G\text{-Zip}^{\mu}$, which is isomorphic to a quotient stacks $[E \backslash G]$, where E is the "zip group" (see [MW04], [Wed99], [PWZ11], [PWZ]). Viehmann and Wedhorn showed in [VW13] that the special fiber $\text{Sh}_{\mathcal{K}}^1$ of any PEL-type Shimura variety admits a universal G -zip of type μ , which gives rise to a faithfully flat morphism of stacks $\zeta : \text{Sh}_{\mathcal{K}}^1 \rightarrow G\text{-Zip}^{\mu}$. The fibers of this morphism are then the Ekedahl-Oort strata. In his thesis ([Zha]), Zhang constructs a universal G -zip in the Hodge-type case, and proves the smoothness of the induced map ζ (see also [Wor]).

1.2.3. *Hasse invariants.* The history of Hasse invariants begins with the example of the modular curve. There exists a section $H \in H^0(X_0(N) \otimes \mathbf{F}_p, \omega^{p-1})$, where ω is the Hodge bundle, whose non-vanishing locus is precisely the ordinary locus of the special fiber of the modular curve (i.e the locus where the elliptic curve satisfies $|E[p](\overline{\mathbf{F}}_p)| = p$). Deligne observed that the Eisenstein series E_{p-1} is a modular form which lifts the Hasse invariant H (for a proof, see [Kat73], 2.1). In the case of Hilbert-type Shimura varieties, Goren defined in [Gor01] partial Hasse invariants (in characteristic p), whose product is the classical one. Under some conditions, he was able to construct Hilbert modular forms in characteristic zero lifting these Hasse invariants. Again in the Hilbert case, Emerton-Reduzzi-Xiao have considered

partial Hasse invariants to attach pseudo-representations to Hecke eigenclasses mod p^n .

For unitary Shimura varieties of signature $(n-1, 1)$ at split primes, Ito constructed Hasse invariants on all closed Ekedahl-Oort strata. Until recently, this was the only nontrivial occurrence of sections on smaller strata. We end this section by mentioning progress made recently on this topic by the first author and Nicole in [GN14] where they define concretely a $(\mu$ -ordinary) Hasse invariant of unitary Shimura varieties. Finally, the second author and Wedhorn extended this result to all Shimura varieties of Hodge-type, by a group-theoretical method. For smaller strata, a partial result by the second author was proved in [Kos14]. An advantage of the approach by G -zips to constructing Hasse invariants is that sections obtained by pull-back from this stack are automatically Hecke-equivariant.

1.3. The history of this work and related preprints/announcements.

1.3.1. *History of this work.* This work began as a joint project of the first author with David Geraghty in the Spring of 2013. In the first draft of this paper, we had previously given a three-sentence account of the first author's point of view concerning the ensuing beginning and ending of George Boxer's collaboration with David Geraghty and the first author. Following helpful discussions with Boxer about what happened, we feel it is important to replace that account with the following:

After the first author began working with Geraghty, the project evolved into a three-way collaboration with George Boxer. In retrospect, it seems that several unfortunate misunderstandings between the three collaborators during the Summer/Fall of 2014 played a pivotal role in the splitting of the collaboration into two separate groups; one consisting Boxer and the other consisting of Geraghty and the first author. In the Summer of 2014, Geraghty decided not to continue further on the project.²

We feel it is best if any analysis of these misunderstandings remains private between the collaborators; in particular the current paper is not the place to enter into such an analysis.

The second author joined this project in the Fall of 2014. His invaluable contributions allowed us to simultaneously sharpen the partial results previously obtained, and to generalize these to the case of Hodge-type Shimura varieties.

1.3.2. *The work of Pilloni-Stroh* [PS]. Pilloni-Stroh announced results similar to those of ours concerning Galois representations around the same time as us. Their preprint [PS] appeared a few months before the first version of this paper was posted on the ArXiv. Let us mention briefly some of the key differences between their work and ours. Their approach is based on Scholze's theory of perfectoid Shimura varieties [Sch], while our approach is more elementary and completely independent of

²Boxer informed Geraghty and the first author on October 13, 2013 via email that he thought he had succeeded in constructing Hasse invariants on all Ekedahl-Oort strata in the Siegel case, but the first author was not provided a proof until after we posted our preprint on the ArXiv on July 17, 2015. As one can tell from the posting dates, it took both Boxer and us roughly 1 1/2 years from that point to post complete proofs of this fact. In an effort to be as transparent as possible, the first author has posted a copy of Boxer's October 13, 2013 email on his webpage, <https://sites.google.com/site/wushijig/>.

Scholze’s work (or any other involved p -adic analysis). This fundamental difference has the following consequences.

The results of Pilloni-Stroh on automorphic representations are somewhat stronger in two ways. First, to produce a p -adic Galois representation, they need not assume the p -adic component of the automorphic representation is unramified, while we do. Second, we have had to make some additional assumptions about the special fiber of our Shimura variety in the general (noncompact, non-PEL) Hodge-type case. However, we believe these assumptions will soon be shown to hold (they are already known in both the PEL and compact cases), so that the latter shortcoming on our part will disappear.

Concerning torsion, our results regard the coherent cohomology of the “modular” integral models of Kottwitz and Vasiu-Kisin, while the results of Pilloni-Stroh concern Scholze’s “strange” integral models. We would argue that the models we study are more natural, because they are moduli spaces of abelian varieties³, they are smooth and their special fibers admit rich stratifications. On the other hand, Scholze’s model satisfies neither one of these three properties.

1.3.3. *The work of Boxer.* The thesis of George Boxer [Box15], [Box] obtains results that both overlap with, and differ from, ours in several respects.⁴ We believe that a more detailed study of the relationship between our two approaches would lead to both a better understanding of both works and to new results which go beyond either approach in itself.

The main idea shared by Boxer’s thesis and this work is that it is possible to construct congruences in the coherent cohomology of certain Shimura varieties by first constructing Hasse invariants on the Ekedahl-Oort strata of these Shimura varieties and then using the Hasse invariants to produce congruences by a novel method. Regarding the construction of Hasse invariants, another idea common to both works is that the problem simplifies if one first works on a larger space mapping down to the Shimura variety in question (see below). Both works were at least partly motivated by applications to the construction of automorphic Galois representations (and torsion generalizations).⁵

There are several important differences between our approach and that of Boxer, both at the level of ideas and at the level of results. We limit ourselves to listing three of the most important ones here. One key difference is that our approach to Hasse invariants is group-theoretical, via the Pink-Wedhorn-Ziegler theory of G -Zips, while Boxer’s approach is based on the canonical filtration, as in Moonen’s

³As is well-known, care must be taken in formulating a precise statement of this kind in the non-PEL case due to the fact that the Hodge Conjecture is not known for abelian varieties.

⁴In the first draft of this paper, we pointed out that we could not comment on Boxer’s work, since no written preprint of his work was available to us. On July 31, after the posting of our preprint, Richard Taylor informed us that it had been possible to view Boxer’s thesis on the Harvard DASH system as of May 18. We were not aware of this and Boxer’s email to us immediately following the posting of our preprint suggests that Boxer himself was unaware that his thesis was publicly accessible before he posted it to the Arxiv on July 21. Before we posted our preprint, we ran several Google searches to see if Boxer — or anyone else for that matter — had publicly posted any written work related to ours and we found nothing besides the aforementioned work of Pilloni-Stroh.

⁵We note that, while Boxer has announced applications of his thesis to the construction of Galois representations, his thesis contains no results about Galois representations. He has informed us that he will soon post a preprint containing applications to Galois representations.

original definition of the Ekedahl-Oort stratification in the PEL case. This difference is one reason why we are able to prove our results for general Hodge-type Shimura varieties, while Boxer’s results are restricted to those Shimura varieties of PEL-type A or C.

Another interesting difference between our approaches is our choice of “larger space” mapping down to the Shimura variety: We use the “flag space”, generalizing the work of Ekedahl-van der Geer in the Siegel case, while Boxer works with a Shimura variety at Iwahori level. Although there is a close relationship between the flag space and the Shimura variety at Iwahori level, this relationship is delicate and not fully understood, and the two objects retain fundamentally different properties. The flag space is a fibration over the Shimura variety with flag variety fibers, while the dimension of the fibers of the Iwahori level Shimura variety over the original (hyperspecial level) Shimura variety is not constant. Moreover, the fibers of the first morphism are connected, while those of the second are not always so. As a consequence, it is (to our knowledge) not currently known how to compare the cohomology “on the top” and “on the bottom” in the Iwahori case, but in our flag space setting we have a natural isomorphism of top and bottom cohomologies by Kempf’s vanishing theorem (for bundles corresponding to dominant weights). In terms of congruences, this allows us to “move the weight” in as many directions as the rank of the character group (see Corollary 3.4.2), while Boxer is only able to move the weight in the “parallel direction” (powers of the Hodge line bundle).

A third notable difference between our works is how we treat compactifications. In this respect, Boxer has obtained much more precise information than us in the PEL case. We show in a rather formal way that the morphism ζ constructed by Zhang from the (special fiber of the) Shimura variety to the stack of G-Zips extends to a morphism ζ^{tor} on a smooth toroidal compactification. Thus we obtain an extension of the Ekedahl-Oort stratification to toroidal compactifications, but we do not give a modular description of the stratification at the boundary. In the PEL case, Boxer goes further by giving a modular extension of the Ekedahl-Oort stratification and describing in detail the relationship between the Ekedahl-Oort and boundary stratifications, generalizing the work of Ekedahl-van der Geer in the Siegel case. This description allows Ekedahl-van der Geer and Boxer to also deduce an extension of the Ekedahl-Oort stratification to the minimal compactification in their respective settings.

1.4. Outline. This paper is structured as follows: §2 introduces notation and collects known results about many of the objects that we consider. More specifically, §2.1 recalls facts about Shimura varieties of Hodge type, their integral models, compactifications, torsors and bundles, coherent cohomology and the structure theory of some of the related reductive groups. Then §2.2 considers those aspects of the representation theory of connected, reductive \mathbf{R} -algebraic groups relevant to us, namely limits of discrete series, their parametrization and Lie algebra cohomology. Hecke algebras and the Satake isomorphism are discussed in §2.3; this serves as the basis for stating our local-global compatibility results for Galois representations and pseudo-representations. Finally, §2.4 introduces conditions on the existence of regular automorphic Galois representations.

Using the notation and conditions introduced in §2, we state our main results in §3. Our main results are naturally divided into five categories: In §3.1, we state the existence of group-theoretical, generalized Hasse invariants on the strata closures

of the stack of G -zips. By pulling back to the Shimura variety, we obtain Hasse invariants on the closures of Ekedahl-Oort strata. These Hasse invariants play an essential role for our other main results. As an immediate corollary, we deduce that all Ekedahl-Oort strata are affine for compact Shimura varieties of Hodge type.

Second, in §3.2 we extend the Ekedahl-Oort stratification to smooth toroidal compactifications. In the Siegel case, we show that the induced strata of the minimal compactification are affine, as conjectured by Oort [Oor01, 14.2]. In §3.4, we describe our results that the action of the Hecke algebra on higher coherent cohomology factors through its action in degree zero. Using these Hecke algebra factorizations, we attach Galois representations to non-degenerate limits of discrete series, and pseudo-representations to torsion in coherent cohomology; for the precise statements, see §3.5. Finally, §3.6 generalizes (part of) Serre's letter to Tate on mod p systems of Hecke eigenvalues to Hodge-type Shimura varieties.

The rest of the paper is devoted to proving the results of §3 and naturally breaks up into three parts. The first, consisting of §§4-6 carries out the construction of Hasse invariants at the level of the stack of G -Zips. The second part, comprised of §§7-9 is a collection of technical results which are needed later on, mostly for proving the results of §3.4. The results stated in §§3.2-3.6 are completed in the third and final part, §§10-11, except for lemma 3.2.1.

2. SHIMURA VARIETIES AND AUTOMORPHIC REPRESENTATIONS

2.1. Shimura varieties of Hodge type.

2.1.1. *The Siegel Shimura datum.* Let $g \in \mathbf{Z}_{\geq 1}$ and let (V, ψ) be a pair consisting of a $2g$ -dimensional \mathbf{Q} -vector space V , equipped with a non-degenerate symplectic form ψ . Let $GSp(2g) = GSp(V, \psi)$ be the connected, reductive \mathbf{Q} -group preserving ψ up to a similitude factor.

Let $\mathbf{S} = \text{Res}_{\mathbf{C}/\mathbf{R}} \mathbf{G}_{m, \mathbf{C}}$ be the Deligne torus. As in [Del77, §1.3.1], let \mathbf{X}_g be the $GSp(2g, \mathbf{R})$ -conjugacy class of homomorphisms $h : \mathbf{S} \rightarrow GSp(2g)_{\mathbf{R}}$ such that

1. The composition $\mathbf{S} \xrightarrow{h} GSp(2g)_{\mathbf{R}} \rightarrow GL(V)$ is a real Hodge structure of type $\{(-1, 0), (0, -1)\}$ on $\mathbf{V} \otimes \mathbf{R}$ for which ψ is a Hodge tensor of type $(1, 1)$
2. The form $\psi(z, h(\mathbf{i})z)$ is definite (either positive or negative).

The pair $(GSp(2g), \mathbf{X}_g)$ is a Shimura datum in the sense of *loc. cit.* which we call a Siegel-type Shimura datum.

2.1.2. *Shimura data of Hodge type.* Let \mathbf{G} be a connected, reductive \mathbf{Q} -algebraic group and let \mathbf{X} be the $\mathbf{G}(\mathbf{R})$ -conjugacy class of a homomorphism

$$h : \mathbf{S} \rightarrow \mathbf{G}_{\mathbf{R}}$$

such that the pair (\mathbf{G}, \mathbf{X}) forms a Shimura datum of Hodge type [Del77, §1.3]. Recall that this means (\mathbf{G}, \mathbf{X}) is a Shimura datum and that there exists an embedding of Shimura data $(\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(2g), \mathbf{X}_g)$ for some $g \in \mathbf{Z}_{\geq 1}$. Write $E = E(\mathbf{G}, \mathbf{X})$ for the reflex field of (\mathbf{G}, \mathbf{X}) , and \mathcal{O}_E for its ring of integers. Given a neat, open compact subgroup $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$, write $Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}$ for Deligne's canonical model at level \mathcal{K} over E (see *loc. cit.*). We denote the resulting tower of E -schemes with its $\mathbf{G}(\mathbf{A}_f)$ action by $Sh(\mathbf{G}, \mathbf{X})$. Define $\dim(\mathbf{G}, \mathbf{X})$ to be the dimension of $Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}$ over E for all $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$.

2.1.3. *Integral models, compactifications.* For every finite prime p , let \mathbf{G}_p be the base change of \mathbf{G} to \mathbf{Q}_p . Let $\text{Ram}(\mathbf{G})$ denote the set of primes where \mathbf{G} is ramified. We assume henceforth that $p \notin \text{Ram}(\mathbf{G})$. This implies that the group $\mathbf{G}(\mathbf{Q}_p)$ admits a hyperspecial subgroup. By a p -hyperspecial level \mathcal{K} , we shall mean an open, compact subgroup of $\mathbf{G}(\mathbf{A}_f)$ of the form $\mathcal{K} = \mathcal{K}^p \mathcal{K}_p$ with \mathcal{K}_p hyperspecial and $\mathcal{K}^p \subset \mathbf{G}(\mathbf{A}_f^p)$.

Let \mathfrak{p} be a prime of E above p and let \mathcal{K}_p be a hyperspecial subgroup of $\mathbf{G}(\mathbf{Q}_p)$. Assume $p > 2$. By the work of Vasiu [Vas99, Th. 0] and Kisin [Kis10, Th. 1], the sub-tower of E -schemes $Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}$, where \mathcal{K} ranges over open compact subgroup of $\mathbf{G}(\mathbf{A}_f)$ of the form $\mathcal{K} = \mathcal{K}_p \mathcal{K}^p$, together with the $\mathbf{G}(\mathbf{A}_f^p)$ action it retains, admits an integral canonical model in the sense of Milne [Mil92] over the completion of \mathcal{O}_E at \mathfrak{p} , which we denote by $\mathcal{O}_{\mathfrak{p}}$. Let $Sh(\mathbf{G}, \mathbf{X}, \mathcal{K}_p)$ (resp. $Sh_{\mathcal{K}}^0$) denote this tower of $\mathcal{O}_{\mathfrak{p}}$ -schemes with its $\mathbf{G}(\mathbf{A}_f^p)$ -action (resp. its component at level \mathcal{K}).

By Chai-Faltings [CF90] (resp. Lan [Lan13], Madapusi [Mad]) in the Siegel (resp. PEL, Hodge) case, one knows that, given certain choices Σ of combinatorial data, there exists a smooth toroidal compactification $Sh_{\mathcal{K}}^{\Sigma, 0}$ of $Sh_{\mathcal{K}}^0$. We shall usually suppress Σ from the notation and write instead $Sh_{\mathcal{K}}^{\text{tor}, 0}$, with the understanding that certain constructions require refining Σ (see §2.1.7). The above works also deduce from the existence of $Sh_{\mathcal{K}}^{\text{tor}, 0}$ a normal integral model of the minimal (Baily-Borel) compactification. We denote the minimal compactification by $Sh_{\mathcal{K}}^{\text{min}, 0}$. The reduction modulo \mathfrak{p}^n of $Sh_{\mathcal{K}}^{\text{tor}, 0}$ (resp. $Sh_{\mathcal{K}}^{\text{min}, 0}$) is denoted $Sh_{\mathcal{K}}^{\text{tor}, n}$ (resp. $Sh_{\mathcal{K}}^{\text{min}, n}$).

We write $D_{\mathcal{K}}$ (resp. $D_{\mathcal{K}}^n$) for the boundary of $Sh_{\mathcal{K}}^{\text{tor}, 0}$ (resp. $Sh_{\mathcal{K}}^{\text{tor}, n}$). By [Mad, Th.1], the boundary $D_{\mathcal{K}}$ is a relative, effective Cartier divisor of $Sh_{\mathcal{K}}^{\text{tor}, 0}$ over $\mathcal{O}_{\mathfrak{p}}$. Since an arbitrary base change of a relative, effective Cartier divisor is again such (cf. [KM85, §1.1.4]), the same holds for $D_{\mathcal{K}}^n$ in $Sh_{\mathcal{K}}^{\text{tor}, n}$ over $\mathcal{O}_E/\mathfrak{p}^n$.

Let

$$(2.1.1) \quad \pi^{\text{tor}, \text{min}} : Sh_{\mathcal{K}}^{\text{tor}, 0} \longrightarrow Sh_{\mathcal{K}}^{\text{min}, 0}$$

denote the canonical morphism between a toroidal compactification and the minimal compactification. Let $\pi_n^{\text{tor}, \text{min}}$ denote the reduction of $\pi^{\text{tor}, \text{min}}$ modulo \mathfrak{p}^n . By the construction of the minimal compactification, one has $\pi_*^{\text{tor}, \text{min}} \mathcal{O}_{Sh_{\mathcal{K}}^{\text{tor}, 0}} = \mathcal{O}_{Sh_{\mathcal{K}}^{\text{min}, 0}}$.

We use a subscript g to denote the objects described above in the case $(\mathbf{G}, \mathbf{X}) = (GSp(2g), \mathbf{X}_g)$. Thus for example $Sh_{g, \mathcal{K}}^0$ denotes the integral model of the Siegel-type Shimura variety for $(GSp(2g), \mathbf{X}_g)$ at level \mathcal{K} and $Sh_{g, \mathcal{K}}^{\text{tor}, 0}$ denotes a smooth, toroidal compactification of it. We denote by $\mathcal{A} \longrightarrow Sh_{g, \mathcal{K}}^0$ the universal abelian scheme ([Del71, 4.11-4.17]).

2.1.4. *Structure theory and root data.* Fix an embedding of Shimura data

$$(2.1.2) \quad \varphi : (\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(2g), \mathbf{X}_g).$$

We can find $h \in \mathbf{X}$ such that the associated minuscule cocharacter μ given by $\mu(z) = (h \otimes \mathbf{C})(z, 1)$ is defined over E . Put $h_g = \varphi \circ h$ and $\mu_g = \varphi \circ \mu$. Let \mathbf{L} (resp. L_g) be the Levi subgroup of \mathbf{G}_E (resp. $GSp(2g)_E$) which stabilizes μ (resp. μ_g). Let P_g be the parabolic subgroup of $GSp(2g)_E$ stabilizing the Hodge filtration of the Hodge structure $\text{Ad} \circ h_g$. Put $\mathbf{P} = \mathbf{G} \cap P_g$; thus \mathbf{P} is a parabolic subgroup of \mathbf{G}_E having \mathbf{L} as Levi factor.

Choose a sextuple $(h, h_g, \mathbf{B}, \mathbf{T}, B_g, T_g)$ (defined over E), where

- Bor1. \mathbf{T} and T_g are maximal tori of \mathbf{G}_E and $GS(2g)_E$ respectively, satisfying the compatibility $T_g \cap \mathbf{G} = \mathbf{T}$ and having the real points compact modulo center.
- Bor2. \mathbf{B} and B_g are Borel subgroups of \mathbf{G}_E and $GS(2g)_E$ respectively, satisfying $B_g \cap \mathbf{G} = \mathbf{B}$.
- Bor3. One has inclusions $\mu(\mathbf{G}_{m,E}) \subset \mathbf{T} \subset \mathbf{B} \subset \mathbf{P}$ and $\mu(\mathbf{G}_m) \subset T_g \subset B_g \subset P_g$.

Write $\mathbf{B}_\mathbf{L} = \mathbf{B} \cap \mathbf{L}$. Then $\mathbf{B}_\mathbf{L}$ is a Borel subgroup of \mathbf{L} satisfying the inclusions $\mathbf{T} \subset \mathbf{B}_\mathbf{L} \subset \mathbf{L}$. We denote the roots of $\mathbf{T}_\mathbf{C}$ in $\mathbf{G}_\mathbf{C}$ (resp. $\mathbf{L}_\mathbf{C}$) by Φ (resp. Φ_c) and write Φ^\vee (resp. Φ_c^\vee) for the corresponding coroots. Put $\Phi_n = \Phi - \Phi_c$. We shall occasionally refer to the elements of Φ_c (resp. $\Phi_c^\vee, \Phi_n, \Phi_n^\vee$) as compact roots (resp. compact coroots, non-compact roots, non-compact coroots), which also suggests the notation. Let Φ^+ be the system of positive roots such that $\alpha \in \Phi^+$ when the $-\alpha$ -root subgroup $U_{-\alpha}$ is contained in $\mathbf{B}(\mathbf{C})$. Put $\Phi_c^+ = \Phi^+ \cap \Phi_c$ and $\Phi_n^+ = \Phi^+ \cap \Phi_n$. Then $\alpha \in \Phi_n^+$ precisely when the α -root space is contained in the $(-1, 1)$ -part of the Hodge structure $\text{Ad} \circ h_g$. Let $\Delta \subset \Phi^+$ (resp. $\Delta_c \subset \Phi_c^+$) be the subset of simple roots and let $\Delta^\vee \subset \Phi^\vee$ (resp. $\Delta_c^\vee \subset \Phi_c^\vee$) be the corresponding subset of simple coroots. Let $\Phi^{\vee,+}$ (resp. $\Phi_c^{\vee,+}$) be the system of positive coroots generated by Δ^\vee (resp. Δ_c^\vee).

Let $X^*(\mathbf{T}_\mathbf{C})$ (resp. $X_*(\mathbf{T}_\mathbf{C})$) denote the group of characters (resp. cocharacters) of $\mathbf{T}_\mathbf{C}$ and write \langle, \rangle for the perfect \mathbf{Z} -pairing between them. The quadruples

$$(2.1.3) \quad BRD = (X^*(\mathbf{T}_\mathbf{C}), \Delta, X_*(\mathbf{T}_\mathbf{C}), \Delta^\vee) \text{ and } BRD_c = (X^*(\mathbf{T}_\mathbf{C}), \Delta_c, X_*(\mathbf{T}_\mathbf{C}), \Delta_c^\vee)$$

are based root data for $\mathbf{G}_\mathbf{C}$ and $\mathbf{L}_\mathbf{C}$ respectively. Let W (resp. W_c) be the Weyl group of BRD (resp. BRD_c). Let w_0 (resp. $w_{0,c}$) be the longest element of W (resp. W_c). We denote by e the identity element of W . Let $X_+^*(\mathbf{T}_\mathbf{C}) \subset X^*(\mathbf{T}_\mathbf{C})$ (resp. $X_{+,c}^*(\mathbf{T}_\mathbf{C}) \subset X^*(\mathbf{T}_\mathbf{C})$) be the cone of Δ^\vee -dominant (resp. Δ_c^\vee -dominant) characters.

2.1.5. Torsors and Bundles. Fix a prime $p \notin \text{Ram}(\mathbf{G})$ and a prime \mathfrak{p} of E above p . Let $\mathbf{L}_\mathfrak{p}$ (resp. $\mathbf{P}_\mathfrak{p}, \mathbf{B}_\mathfrak{p}, (\mathbf{B}_\mathbf{L})_\mathfrak{p}$) denote the base change of \mathbf{L} (resp. $\mathbf{P}, \mathbf{B}, \mathbf{B}_\mathbf{L}$) to $E_\mathfrak{p}$. Since $p \notin \text{Ram}(\mathbf{G})$, the \mathbf{Q}_p -group \mathbf{G}_p extends to a smooth reductive \mathbf{Z}_p -group scheme, which we denote G^0 . The conjugacy class $[\mu]$ extends to a conjugacy class of cocharacters of G^0 defined over $\mathcal{O}_\mathfrak{p}$, and we may choose a representative μ defined over $\mathcal{O}_\mathfrak{p}$ since G^0 is quasi-split. This defines a model L^0 of $\mathbf{L}_\mathfrak{p}$ over $\mathcal{O}_\mathfrak{p}$. Since \mathbf{Z}_p and $\mathcal{O}_\mathfrak{p}$ are discrete valuation rings, [ABD⁺66, Chap.XXVI, 3.5] (coupled with the valuative criterion of properness) implies that the $E_\mathfrak{p}$ -subgroups $\mathbf{P}_\mathfrak{p}$ and $\mathbf{B}_\mathfrak{p}$ extend uniquely to parabolic $\mathcal{O}_\mathfrak{p}$ -subgroups. We denote these $\mathcal{O}_\mathfrak{p}$ -subgroups by P^0, B^0 respectively, and we set $B_L^0 := L^0 \cap B^0$, suppressing \mathfrak{p} from the notation. A superscript n in place of 0 will denote the reduction of the $\mathcal{O}_\mathfrak{p}$ -group modulo \mathfrak{p}^n .

The integral model $\text{Sh}_\mathcal{K}^{\text{tor},0}$ is canonically endowed with torsors $\mathfrak{L}^0, \mathfrak{P}^0, \mathfrak{G}^0$ for the groups $L^0 \subset P^0 \subset G^0$, respectively. We define the G^0 -torsor \mathfrak{G}^0 as in [Mad, Proposition 5.3.2]. Recall that, by [Kis10, Proposition 1.3.2], G^0 is the stabilizer of a collection $\{s_\alpha\}$ of tensors in $V_{\mathbf{Z}_p}^\otimes$, where $V_{\mathbf{Z}_p}$ is a symplectic lattice in $V \otimes \mathbf{Q}_p$ (§2.1.1). Corresponding to the tensors $\{s_\alpha\}$, Kisin⁶ [Kis10, 2.3.9] constructs global sections of $\varphi^* H_{1,\text{dR}}(\mathcal{A}_g/\text{Sh}_{g,\mathcal{K}}^0)^\otimes$ and [Mad, Proposition 5.3.2] shows that these

⁶Note that Kisin uses de Rham cohomology while Madapusi Pera uses de Rham homology.

sections extend to sections

$$(2.1.4) \quad s_{\alpha, \text{dR}} \in H^0(\text{Sh}_{\mathcal{K}}^{\text{tor}, 0}, \varphi^* H_{1, \text{dR}}^{\text{can}}(\mathcal{A}_g / \text{Sh}_{g, \mathcal{K}}^0)^{\otimes}),$$

where the superscript “can” signifies the canonical extension (with logarithmic poles along the boundary) to a toroidal compactification $\text{Sh}_{g, \mathcal{K}}^{\text{tor}, 0}$. The G^0 -torsor \mathfrak{G}^0 is now defined as the torsor of bases respecting the s_{α} and the $s_{\alpha, \text{dR}}$.

The P^0 -torsor \mathfrak{P}^0 is defined in Prop. 5.3.4 of *loc. cit.* We define the L^0 -torsor \mathfrak{L}^0 as the quotient $\mathfrak{P}^0 / R_u(P^0)$, where $R_u(\cdot)$ denotes the unipotent radical. We define $L^n \subset P^n \subset G^n$ and $\mathfrak{L}^n, \mathfrak{P}^n, \mathfrak{G}^n$ by reduction modulo \mathfrak{p}^n .

Let N be a finite extension of $E_{\mathfrak{p}}$ with ring of integers \mathcal{O}_N and prime \wp lying over \mathfrak{p} . Since $\mathfrak{P}^0 / P^0 \cong \text{Sh}_{\mathcal{K}}^{\text{tor}, 0}$, applying the standard “associated sheaves” construction (cf. [Jan03, §5.8]) to a representation of P^0 on a free \mathcal{O}_N -module W gives rise to a vector bundle on $\text{Sh}_{\mathcal{K}}^{\text{tor}, 0}$ of the same rank as that of W over $\mathcal{O}_{\mathfrak{p}}$. By setting $R_u(P^0)$ to act trivially, any representation of L^0 on a free $\mathcal{O}_{\mathfrak{p}}$ -module, gives rise to one of P^0 .

Let $\eta \in X_{+, c}^*(\mathbf{T}_{\mathbf{C}})$. Let \mathcal{L}_{η} be the $\mathbf{L}_{\mathbf{C}}$ -equivariant (or $\mathbf{L}_{\mathbf{C}}$ -linearized) line bundle on the flag variety $\mathbf{L}_{\mathbf{C}} / (\mathbf{B}_{\mathbf{L}})_{\mathbf{C}}$ associated to the character η again by the “associated sheaves” construction. Then there exists an extension N as above such that \mathcal{L}_{η} descends to an L^0 -equivariant line bundle on $L^0 / B_L^0 \times_{\mathcal{O}_{\mathfrak{p}}} \mathcal{O}_N$ over \mathcal{O}_N . We continue to call the descended line bundle \mathcal{L}_{η} . Let V_{η} be the representation of L^0 on $H^0(L^0 / B_L^0 \times \mathcal{O}_N, \mathcal{L}_{\eta})$. One knows by the Borel-Weil part of the Borel-Weil-Bott theorem that $V_{\eta} \otimes \mathbf{C}$ is irreducible of highest weight η . Let $\mathcal{V}_{\eta}^{\text{can}}$ be the vector bundle on $\text{Sh}_{\mathcal{K}}^{\text{tor}, 0}$ (defined over \mathcal{O}_N) constructed from the torsor \mathfrak{L}^0 and the representation V_{η} .

Let \mathcal{V}_{η} be the restriction of $\mathcal{V}_{\eta}^{\text{can}}$ to $\text{Sh}_{\mathcal{K}}^0$ and put $\mathcal{V}_{\eta}^{\text{sub}} = \mathcal{V}_{\eta}^{\text{can}}(-D_{\mathcal{K}})$. Since $D_{\mathcal{K}}$ is a relative, effective Cartier divisor, $\mathcal{V}_{\eta}^{\text{sub}}$ is again a vector bundle. As the level \mathcal{K} varies, the system of vector bundles \mathcal{V}_{η} (resp. $\mathcal{V}_{\eta}^{\text{can}}, \mathcal{V}_{\eta}^{\text{sub}}$) so obtained forms a $\mathbf{G}(\mathbf{A}_f^p)$ -equivariant vector bundle: For every $g \in \mathbf{G}(\mathbf{A}_f^p)$ and every \mathcal{K} , there exist choices Σ, Σ' of combinatorial data for toroidal compactifications such that we have a canonical isomorphism between the bundle at level $g^{-1}\mathcal{K}g$ and at level \mathcal{K} , induced by the isomorphism $\text{Sh}_{\mathcal{K}}^{\text{tor}, 0} \xrightarrow{\sim} \text{Sh}_{g^{-1}\mathcal{K}g}^{\text{tor}, 0}$.

We call the bundle \mathcal{V}_{η} (resp. $\mathcal{V}_{\eta}^{\text{can}}, \mathcal{V}_{\eta}^{\text{sub}}$) the automorphic vector bundle of highest weight η (resp. its canonical extension, subcanonical extension), because this bundle is a generalization to the Hodge case of those studied (and so named) by Lan (cf. [Lan13], [Lan]) in the PEL case.

2.1.6. The Hodge line bundle. We define the Hodge line bundle $\omega(\varphi)$ on $\text{Sh}_{\mathcal{K}}^{\text{tor}, 0}$, associated to the symplectic embedding φ (see (2.1.2)), as in [Mad, Definition 5.1.2]. In the Siegel case, set

$$(2.1.5) \quad \Omega_g = \text{Fil}^1 H_{\text{dR}}^1(\mathcal{A}_g / \text{Sh}_{g, \mathcal{K}}^0) \text{ and } \omega_g = \det \Omega_g.$$

In general put $\Omega(\varphi) = \varphi^* \Omega_g$ (resp. $\Omega(\varphi) = \varphi^* \Omega_g$). Note that both $\Omega(\varphi)$ and its determinant $\omega(\varphi)$ depend on the embedding φ ; neither is, strictly speaking, an intrinsic invariant of the Shimura datum⁷. However, we shall usually fix a single embedding φ , and then drop φ from the notation, writing simply ‘ ω ’.

⁷For example, let $(\mathbf{G}, \mathbf{X}) = (GSp(2), \mathbf{X}_1) = (GL(2), \mathbf{X}_1)$ (the case of modular curves), let $\varphi_1 : (GSp(2), \mathbf{X}_1) \rightarrow (GSp(2), \mathbf{X}_1)$ be the identity and let $\varphi_2 : (GSp(2), \mathbf{X}_1) \rightarrow (GSp(4), \mathbf{X}_2)$ be the “diagonal” embedding. Then $\omega(\varphi_2) = (\omega(\varphi_1))^2$.

Write $\text{Std} : GSp(V, \psi) \hookrightarrow GL(V)$ for the forgetful representation (notation as in §2.1.1). Let $\eta_{g, \text{Std}}$ be the highest weight of Std . Then we have $\mathcal{V}_{\eta_{g, \text{Std}}}^\vee \cong \Omega_g$.

Definition 2.1.1 (quasi-constant characters). *Let $\chi \in X^*(\mathbf{T})_{\mathbf{Q}}$. We say that χ is quasi-constant on Weyl-Galois orbits (or simply quasi-constant for short) if, for every coroot α^\vee satisfying $\langle \chi, \alpha^\vee \rangle \neq 0$ and all $\sigma \in W \rtimes \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we have either $|\langle \chi, \alpha^\vee \rangle| = |\langle \chi, \sigma \alpha^\vee \rangle|$ or $\langle \chi, \sigma \alpha^\vee \rangle = 0$.*

In other words, χ is quasi-constant if, for every $W \rtimes \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -orbit of coroots O , the set of absolute values of pairings $\{|\langle \chi, \alpha^\vee \rangle| \mid \alpha^\vee \in O\}$ is either a singleton, or it consists of two elements, one of which is zero.

Remark 2.1.2. It is immediate from the definition that any scalar multiple of a minuscule character is quasi-constant. On the other hand, the largest fundamental weight in a root system of type C (resp. the smallest fundamental weight in a root system of type B) is an example of a quasi-constant character which is not a multiple of a minuscule character. So the notion of quasi-constant character may be seen as a non-trivial generalization of the notion of minuscule character.

In Appendix A, we shall prove the following theorem, which will be used to show that our main result on group-theoretical Hasse invariants (see Theorem 3.1.2) applies to all Shimura varieties of Hodge type (Corollary 3.1.3).

Theorem 2.1.3. *For every symplectic embedding $\varphi : (\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(2g), \mathbf{X}_g)$, the character η_ω of the Hodge line bundle is quasi-constant on Weyl-Galois orbits.*

Remark 2.1.4. In the PEL-case (when φ is the "forgetful" map), it is well-known how to write down the character η_ω explicitly, and then to check case-by-case that Theorem 2.1.3 holds. Below, we give a more conceptual proof which works in the general Hodge case.

2.1.7. *Coherent cohomology.* We define the admissible $\mathbf{G}(\mathbf{A}_f)$ -module

$$(2.1.6) \quad \bar{H}^i(Sh(\mathbf{G}, \mathbf{X}), \mathcal{V}_\eta)$$

as in [Har88, §2]; this definition is recalled in [Tay91, §3.2]. We recall that first, for all $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$, one sets $\bar{H}^i(Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}^\Sigma, \mathcal{V}_\eta)$ to be the image of $H^i(Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}^\Sigma, \mathcal{V}_\eta^{\text{sub}})$ in $H^i(Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}^\Sigma, \mathcal{V}_\eta^{\text{can}})$. Second, $\bar{H}^i(Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}, \mathcal{V}_\eta)$ is defined as a direct limit over refinements $\Sigma' \rightarrow \Sigma$ of $\bar{H}^i(Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}^\Sigma, \mathcal{V}_\eta)$. Finally, $\bar{H}^i(Sh(\mathbf{G}, \mathbf{X}), \mathcal{V}_\eta)$ is defined as the direct limit of $\bar{H}^i(Sh(\mathbf{G}, \mathbf{X})_{\mathcal{K}}, \mathcal{V}_\eta)$ over \mathcal{K} . Cohomology at level \mathcal{K} is recovered from $\bar{H}^i(Sh(\mathbf{G}, \mathbf{X}), \mathcal{V}_\eta)$ by taking \mathcal{K} -invariants.

2.2. Archimedean representation theory.

2.2.1. *Harish-Chandra isomorphism and regularity conditions.* Let $\rho \in X^*(\mathbf{T}_{\mathbf{C}}) \otimes \mathbf{Q}$ be the half-sum of the positive roots with respect to Φ^+ . Following [BG11], we say that $\chi \in X^*(\mathbf{T}_{\mathbf{C}}) \otimes \mathbf{C}$ is L -algebraic (resp. C -algebraic) if $\chi \in X^*(\mathbf{T}_{\mathbf{C}})$ (resp. $\chi \in X^*(\mathbf{T}_{\mathbf{C}}) + \rho$).

Let \mathfrak{g} be the complexified Lie algebra of \mathbf{G} and let \mathfrak{z} be the center of the universal enveloping algebra of \mathfrak{g} . We normalize the Harish-Chandra isomorphism

$$(2.2.1) \quad \mathfrak{z} \xrightarrow{\sim} (X^*(\mathbf{T}_{\mathbf{C}}) \otimes \mathbf{C})^W$$

in the usual way, meaning that the infinitesimal character of the trivial representation is identified with ρ .

Let A be the classical topology connected component of the maximal \mathbf{R} -split torus of the center of $\mathbf{G}_{\mathbf{R}}$. Let K be a maximal compact subgroup of $\mathbf{G}(\mathbf{R})$. For the parametrization of limits of discrete series in §2.2.2, we choose K so that $\mathbf{T}(\mathbf{R}) \cap K$ is a maximal torus in K and $\mathbf{T}(\mathbf{R})$ is the direct product of A with the centralizer $\text{Cent}_{\mathbf{T}(\mathbf{R}) \cap K}$.

Let π_{∞} be an irreducible, admissible $(\mathfrak{g}_{\mathbf{C}}, K_{\mathbf{C}})$ -module, which refer to simply as a Harish-Chandra module. We write χ_{∞} for the infinitesimal character of π_{∞} , identified with an element of $X^*(\mathbf{T}_{\mathbf{C}})/W$ via (2.2.1). Given a non-negative real number δ , and a subset $\Psi^{\vee} \subset \Phi^{\vee}$, we say that $\eta \in X^*(\mathbf{T}_{\mathbf{C}})$ is (δ, Ψ^{\vee}) -regular if

$$(2.2.2) \quad |\langle \chi, \alpha^{\vee} \rangle| > \delta \text{ for all } \alpha^{\vee} \in \Psi^{\vee}.$$

When $\Psi^{\vee} = \Phi^{\vee}$ we say simply δ -regular instead of (δ, Φ^{\vee}) -regular. Thus the standard notion of “regular for $\mathbf{G}_{\mathbf{C}}$ ” corresponds to 0-regular. Given a set of characters, we use a superscript reg (resp. $-\delta$) to denote the subset of regular (resp. δ -regular ones).

Given a property \mathcal{P} of elements of $X^*(\mathbf{T}_{\mathbf{C}}) \otimes \mathbf{C}$ which is stable under the action of W , we say that π_{∞} has property \mathcal{P} if χ_{∞} does.

2.2.2. Limits of discrete series. A full parametrization of limits of discrete series was given by one of us in both [Gol11b] and [Gol11a], based on explanations by Schmid and Vogan ⁸.

We shall review only the part of the parametrization concerning \mathbf{C} -algebraic limits of discrete series, since we shall only be concerned with them in this paper. (By \mathbf{C} -algebraic limit of discrete series, we mean simply a Harish-Chandra module which is both \mathbf{C} -algebraic and a limit of discrete series.) For a fuller picture concerning possible “twisting” of these, we refer to *loc. cit.*

Put $T_K = \mathbf{T}(\mathbf{R}) \cap K$. Define a Weyl group W_K by $\mathbf{N}_K(T_K)/\text{Cent}_K(T_K)$, where $\mathbf{N}_K(T_K)$ is the normalizer of T_K in K . We naturally have $W_c \hookrightarrow W_K$ and W_K acts on $X^*(\mathbf{T}_{\mathbf{C}}) \otimes \mathbf{C}$. If $\mathbf{G}(\mathbf{R})$ is connected in the classical topology, then $W_c = W_K$, but otherwise W_K is bigger.

A \mathbf{C} -algebraic, discrete series Harish-Chandra parameter is an element of

$$(X^*(\mathbf{T}_{\mathbf{C}}) + \rho)^{\text{reg}}.$$

Associated to $\lambda \in (X^*(\mathbf{T}_{\mathbf{C}}) + \rho)^{\text{reg}}$, there is a discrete series Harish-Chandra module π_{λ} with infinitesimal character the orbit $W\lambda$. Moreover, the map $\lambda \mapsto \pi_{\lambda}$ induces a bijection

$$(2.2.3) \quad (X^*(\mathbf{T}_{\mathbf{C}}) + \rho)^{\text{reg}}/W_K \xrightarrow{\sim} \left\{ \begin{array}{c} \mathbf{C}\text{-algebraic discrete series} \\ \text{Harish-Chandra} \\ \text{modules} \end{array} \right\} / \cong$$

By a result of Salamanca-Riba [SR98, Th. 1.8] on the cohomologically induced $A_{\mathfrak{q}}(\lambda)$ -modules, if $\lambda \in X^*(\mathbf{T}_{\mathbf{C}}) + \rho$ is Δ^{\vee} -dominant and ρ -regular, then every unitary Harish-Chandra module of infinitesimal character $W\lambda$ is a discrete series.

Let \mathcal{C} be an (open) Weyl chamber for Φ^{\vee} with closure $\overline{\mathcal{C}}$ and boundary $\partial\mathcal{C} = \overline{\mathcal{C}} - \mathcal{C}$. A \mathbf{C} -algebraic limit of discrete series Harish-Chandra parameter is a pair (λ, \mathcal{C}) , where $\lambda \in \partial\mathcal{C} \in \cap X^*(\mathbf{T}_{\mathbf{C}}) + \rho$ and $\langle \lambda, \alpha^{\vee} \rangle \neq 0$ for all compact \mathbf{C} -simple

⁸Note that most references assume either that $\mathbf{G}(\mathbf{R})$ is connected in the classical topology, or that \mathbf{G} is semisimple; in most cases our group \mathbf{G} satisfies neither of these conditions. Moreover, some references, notably [Har88], state theorems without these assumptions that are only valid with them.

coroots α^\vee (compare [KZ82]). The Weyl group W_K acts on such parameters componentwise, i.e., $w(\lambda, \mathcal{C}) = (w\lambda, w\mathcal{C})$ for all $w \in W_K$. For every \mathcal{C} -algebraic limit of discrete series Harish-Chandra parameter (λ, \mathcal{C}) , there exists a \mathcal{C} -algebraic limit of discrete series Harish-Chandra module $\pi(\lambda, \mathcal{C})$. The Harish-Chandra module $\pi(\lambda, \mathcal{C})$ may be constructed as the image of a discrete series $\pi_{\lambda'}$ by the Zuckerman translation functor $\text{Transl}_{W\lambda}$ which goes from Harish-Chandra modules of infinitesimal character $W\rho$ to ones of infinitesimal character $W\lambda$ (cf. the discussion preceding Th. 2.1 in [Soe97]).

Generalizing (2.2.3), the map $(\lambda, \mathcal{C}) \mapsto \pi(\lambda, \mathcal{C})$ induces a bijection

$$(2.2.4) \quad \left\{ (\lambda, \mathcal{C}) \left| \begin{array}{l} \lambda \in X^*(\mathbf{T}_{\mathbf{C}}) + \rho \cap \partial\mathcal{C}, \\ \langle \lambda, \alpha^\vee \rangle \neq 0 \\ \text{for all } \mathcal{C}\text{-simple } \alpha^\vee \in \Phi_c^\vee \end{array} \right. \right\} / W_K \xrightarrow{\sim} \left\{ \begin{array}{l} \mathcal{C}\text{-algebraic LDS} \\ \text{Harish-Chandra} \\ \text{modules} \end{array} \right\} / \cong$$

The fundamental dichotomy of non-degenerate versus degenerate was introduced for limits of discrete series by Knapp-Zuckerman in their classification of tempered representations [KZ82]. A limit of discrete series $\pi(\lambda, \mathcal{C})$ is degenerate if λ is orthogonal to a compact root (which is then necessarily not \mathcal{C} -simple). If $\pi(\lambda, \mathcal{C})$ is not degenerate, it is called non-degenerate.

Following [Gol14], we say that a limit of discrete series $\pi(\lambda, \mathcal{C})$ is \mathbf{X} -holomorphic if $w\lambda$ is Δ^\vee -dominant for some $w \in W_K$ (here \mathbf{X} pertains to the Shimura datum (\mathbf{G}, \mathbf{X})). As noted in *loc. cit.*, every \mathbf{X} -holomorphic limit of discrete series is non-degenerate.

One of us has shown [Gol11a] that an equivalent formulation of degeneracy is that the image of the restriction to \mathbf{C}^\times of the Langlands parameter of $\pi(\lambda, \mathcal{C})$ contains a simple group of rank at least two. In turn, this characterization allows to generalize the non-degenerate/degenerate dichotomy to arbitrary Harish-Chandra modules, ([Gol15], [Gol]).

2.2.3. Lie algebra cohomology. Let \mathfrak{p} (resp. \mathfrak{l} , \mathfrak{b} , \mathfrak{t}) be the complexified Lie algebra of \mathbf{P} (resp. \mathbf{L} , \mathbf{B} , \mathbf{T}). Let \mathfrak{b}' (resp. \mathfrak{n}') be any Borel subalgebra of \mathfrak{g} containing \mathfrak{t} (resp. its maximal nilpotent subalgebra). For $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$, let V_η be the finite-dimensional representation of \mathfrak{l} of highest weight η (so we are committing the abuse of writing V_η for the differential of the complexification of the representation that was called V_η in §2.1.5). Given a Harish-Chandra module π_∞ for $\mathbf{G}(\mathbf{R})$ and $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$, we write $H^i(\mathfrak{p}, \mathfrak{l}, \pi_\infty \otimes V_\eta)$ for the relative Lie algebra cohomology of the pair $(\mathfrak{p}, \mathfrak{l})$ with coefficients in $\pi_\infty \otimes V_\eta$ (cf. [VZ84], [Gui80], [BW80], though note that the latter reference's notation is somewhat different).

The \mathfrak{n}' -cohomology $H^i(\mathfrak{n}, \pi_\infty \otimes \eta)$ of discrete series π_∞ was computed by Schmid [Sch76]. The computation was generalized to non-degenerate limits of discrete series by Williams [Wil88b]. Although we shall not use it, we remark that Soergel gave an algorithm to compute the \mathfrak{n}' -cohomology of degenerate limits of discrete series [Soe97]. Soergel's result implies that the $(\mathfrak{p}, \mathfrak{l})$ -cohomology of a degenerate limit of discrete series is identically zero, which reproves an unpublished result of Mirkovic [Mir88].

The computation of \mathfrak{n}' -cohomology is equivalent to that of $(\mathfrak{b}', \mathfrak{t})$ -cohomology. In turn, $(\mathfrak{b}, \mathfrak{t})$ -cohomology is closely related to $(\mathfrak{p}, \mathfrak{l})$ cohomology. On the one hand, there is a Hochschild-Serre spectral sequence relating the two, which is the relative

Lie algebra cohomology analogue of the Leray spectral sequence for the map $\pi_0^{\text{Fl, Sh}} : \text{Fl}_{\mathcal{K}}^0 \rightarrow \text{Sh}_{\mathcal{K}}^0$ which will be defined in §10.3.1.

On the other hand, the method of Schmid-Williams can be transposed from \mathfrak{n}' to $(\mathfrak{p}, \mathfrak{l})$ -cohomology, as was observed by Harris [Har88, Theorem 3.4]. Unfortunately, Harris made the mistake alluded to in Footnote 8 (in §2.2.2) of stating the result for a group of the type $\mathbf{G}(\mathbf{R})$, while part of the result is only true under the additional hypothesis that $\mathbf{G}(\mathbf{R})$ is connected in the classical topology. (Some of the notation leading up to the result is also inappropriate for the non-semi-simple case, but that bit is easily rectified.) More precisely, the part of Theorem 3.4 of *loc. cit.* which asserts the existence of one-dimensional cohomology is true. The part which says that the cohomology in all other degrees and weights is zero is false, as is already the case for $GL(2)$.

We shall only use the first, correct part. Namely, we have the following:

Theorem 2.2.1 (Schmid-Williams-Harris). *Let $\pi(\lambda, \mathbf{C})$ be a non-degenerate limit of discrete series (§2.2.2), normalized by requiring that λ be Δ_c^\vee -dominant and regular. Then*

$$(2.2.5) \quad \dim H^{\text{cd}(\mathcal{C})}(\mathfrak{p}, \mathfrak{l}, \pi(\lambda, \mathcal{C}) \otimes V_{-w_{0,c}\lambda-\rho}) = 1,$$

where $\text{cd}(\mathcal{C})$, the cohomology degree of \mathcal{C} is defined by

$$(2.2.6) \quad \text{cd}(\mathcal{C}) = \text{Card}(\{\alpha^\vee \in \Phi^{\vee,+} \mid \langle \mathcal{C}, \alpha^\vee \rangle < 0\}).$$

Combining the above theorem with Theorem 2.7, Proposition 3.2.2, and Formula 3.0.2 of [Har88], one deduces:

Corollary 2.2.2. *Suppose $\pi = \pi_f \otimes \pi(\lambda, \mathcal{C})$ is a cuspidal automorphic representation of \mathbf{G} , with λ normalized to be Δ_c^\vee -dominant and regular. Then there is a $\mathbf{G}(\mathbf{A}_f)$ -equivariant embedding*

$$(2.2.7) \quad \pi_f \hookrightarrow \bar{H}^{\text{cd}(\mathcal{C})}(Sh(\mathbf{G}, \mathbf{X}), \mathcal{V}_{-w_{0,c}\lambda-\rho}).$$

2.3. Hecke algebras and the Satake isomorphism. Fix once and for all an isomorphism $\iota : \overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$.

2.3.1. Hecke algebras. Fix $p \notin \text{Ram}(\mathbf{G})$. For every $v \notin \text{Ram}(\mathbf{G}) \cup \{p, \infty\}$, let \mathcal{K}_v be a hyperspecial subgroup of $\mathbf{G}(\mathbf{Q}_v)$. Let $\mathcal{H}_v = \mathcal{H}_v(\mathbf{G}_v, \mathcal{K}_v; \mathbf{Z}_p)$ be the unramified (or spherical) Hecke algebra of \mathbf{G} at v , with \mathbf{Z}_p -coefficients, normalized by the unique Haar measure which assigns the hyperspecial subgroup \mathcal{K}_v volume 1. Put

$$(2.3.1) \quad \mathcal{H} = \mathcal{H}(\mathbf{G}) = \bigotimes_{v \notin S} \mathcal{H}_v \quad (\text{restricted tensor product}).$$

For every quadruple $(i, n, \eta, \mathcal{K})$ consisting of $i, n \in \mathbf{Z}_{\geq 0}$, $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ and a p -hyperspecial level \mathcal{K} , let $\mathcal{H}_{\mathcal{K}}(i, n, \eta)$ denote the image of \mathcal{H} in $\text{End}(H^i(\text{Sh}_{\mathcal{K}}^{\text{tor}, n}, \mathcal{V}^{\text{sub}}))$. An arrow $\mathcal{H} \rightarrow \mathcal{H}_{\mathcal{K}}(i, n, \eta)$ will always signify the defining projection.

2.3.2. *Applications of the Satake isomorphism.* Let ${}^L\mathbf{G} = {}^L\mathbf{G}^\circ \rtimes W_{\mathbf{Q}}$ denote the L-group of \mathbf{G} , defined as in [Bor79, §2], except that we replace the Galois group with the Weil group, see Rmk. 2.4 (2) of *loc. cit.* Given a finite-dimensional, complex, algebraic representation $r : {}^L\mathbf{G} \rightarrow GL(m)$, we denote by $r_v : {}^L\mathbf{G}_v \rightarrow GL(m)$ the representation of the same type, but of ${}^L\mathbf{G}_v$, obtained by restriction.

Let $R_{\text{fd}}({}^L\mathbf{G}_v)$ denote the subalgebra of the group algebra $\mathbf{C}[{}^L\mathbf{G}_v]$ generated by the characters of finite-dimensional, complex, algebraic representations of ${}^L\mathbf{G}_v$. Let $R_{\text{fd}}^{\text{ss}}({}^L\mathbf{G}_v)$ be the algebra obtained by considering elements of $R_{\text{fd}}({}^L\mathbf{G}_v)$ as functions on the set of ${}^L\mathbf{G}_v^\circ$ -conjugacy classes of ${}^L\mathbf{G}_v$ and then restricting to the subset of semi-simple ${}^L\mathbf{G}_v^\circ$ -conjugacy classes. As explained in §§6-7 of *loc. cit.*, composing the Satake isomorphism with the isomorphism of Prop. 6.7 of *loc. cit.* gives a canonical identification⁹

$$(2.3.2) \quad \mathcal{H}_v \xrightarrow{\sim} R_{\text{fd}}^{\text{ss}}({}^L\mathbf{G}_v) .$$

We state two immediate, well-known consequences of (2.3.2) which will play a crucial role in our characterizations of the Galois representations and pseudo-representations that we shall construct.

The first is that there is a bijection between the set of (complex) characters of \mathcal{H}_v and the set of ${}^L\mathbf{G}_v^\circ$ -conjugacy classes of semi-simple elements in ${}^L\mathbf{G}_v$. Since the former set is in bijection with the set of unramified, admissible, complex representations of \mathbf{G}_v , so is the latter. If π_v is an unramified, admissible, complex representation of \mathbf{G}_v , we denote by $\text{Class}(\pi_v)$ the corresponding ${}^L\mathbf{G}_v^\circ$ -conjugacy class. In addition, let $\text{Class}(\pi_v, r_v)$ be the image-induced conjugacy class in $GL(m, \mathbf{C})$ and let $\text{Class}_{p,\iota}(\pi_v, r_v)$ denote the conjugacy class in $GL(n, \overline{\mathbf{Q}}_p)$ obtained from $\text{Class}(\pi_v, r_v)$ by means of the isomorphism ι .

Second, for all $j \in \mathbf{Z}_{\geq 1}$ the function

$$(2.3.3) \quad \begin{aligned} {}^L\mathbf{G}_v &\xrightarrow{\text{tr}^j(r)} \mathbf{C} \\ \tilde{g} &\longmapsto \text{tr}(r(\tilde{g})^j) \quad (\text{for all } \tilde{g} \in {}^L\mathbf{G}_v) \end{aligned}$$

lies in $R_{\text{fd}}^{\text{ss}}({}^L\mathbf{G}_v)$. Using the canonical isomorphism (2.3.2), one has an element of \mathcal{H}_v associated to $\text{tr}^j(r)$; we denote this Hecke operator by $T_v^{(j)}(r)$.

Write $T_v^{(j)}(r; i, n, \eta, \mathcal{K})$ for the image of $T_v^{(j)}(r)$ in $\mathcal{H}_{\mathcal{K}}(i, n, \eta)$.

2.4. Existence of regular Galois representations. We formulate very weak conditions on the existence of Galois representations associated to automorphic representations with “extremely regular” discrete series archimedean component. Under these conditions, we shall be able to construct Galois representations for non-degenerate limit of discrete series archimedean component. Let π be an C -algebraic automorphic representation of \mathbf{G} . We set $\text{Ram}(\pi)$ to be the set of places where π is ramified and put $\text{Ram}(\mathbf{G}, \pi) = \text{Ram}(\mathbf{G}) \cup \text{Ram}(\pi)$. Given a prime $p \notin \text{Ram}(\mathbf{G}, \pi)$, and a finite-dimensional, complex, algebraic representation $r : {}^L\mathbf{G} \rightarrow GL(m)$, we say that the pair (π, r) admits a p -adic Galois representation with weak local-global compatibility, or for short that it satisfies $[\text{GalRep-}p]$, if it satisfies the following condition:

⁹The algebra we call $R_{\text{fd}}^{\text{ss}}({}^L\mathbf{G}_v)$ is denoted A in Prop.6.7 of *loc. cit.*

Condition 2.4.1 (GalRep- p). *There exists a unique semisimple Galois representation*

$$(2.4.1) \quad R_{p,\iota}(\pi, r) : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow GL(m, \overline{\mathbf{Q}}_p)$$

such that, for every place $v \notin \text{Ram}(\mathbf{G}, \pi)$ with $v \neq p$, one has an equality of conjugacy classes $R_{p,\iota}(\pi, r)(\text{Frob}_v) = \text{Class}_{p,\iota}(\pi_v, r_v)$, where Frob_v is a geometric Frobenius element at v .

We want conditions that say that Condition 2.4.1 holds, if only for a very distinguished set of π . Our first condition is quite technical, but we hope it will be useful in applications. We follow it with a stronger condition which is simpler to state, but possibly harder to prove.

Let $h_{w_{0,M}} : X^*(\mathbf{T}_{\mathbf{C}}) \rightarrow X^*(\mathbf{T}_{\mathbf{C}})$ be the function defined by (6.1.3) for the element $w_{0,M}$, where $w_{0,M}$ is the longest element of W_M defined in §5.3. For now, we remark that $h_{w_{0,M}} \otimes \mathbf{Q}$ is an isomorphism which maps the cone of Δ^\vee -dominant characters into the cone of Δ_c^\vee -dominant characters. See Proposition 6.4.1 for a sub-cone contained in the image of $h_{w_{0,M}}$.

Consider the following two conditions on a triple $(\mathbf{G}, \mathbf{X}, r)$, where (\mathbf{G}, \mathbf{X}) is a Hodge-type Shimura datum and $r : {}^L\mathbf{G} \longrightarrow GL(m)$ is as before:

Condition 2.4.2 (ERG- p). *Let $p \notin \text{Ram}(\mathbf{G})$. Then for every $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$, there exists a character χ_η contained in the image of $X_+^*(\mathbf{T}_{\mathbf{C}})$ under the map $h_{w_{0,M}}$, which satisfies the following: For all $k \in \mathbf{Z}_{\geq 1}$, there exists $a_0(k) \in \mathbf{Z}_{\geq 1}$ such that, for all $a \geq a_0(k)$, one has*

- (1) *The character $-w_{0,c}(\eta + a\eta_\omega + k\chi_\eta)$ is Δ^\vee -dominant and Φ^\vee -regular.*
- (2) *Every cuspidal automorphic representation π of \mathbf{G} whose archimedean component is the discrete series with Harish-Chandra parameter*

$$(2.4.2) \quad -w_{0,c}(\eta + a\eta_\omega + k\chi_\eta + \rho)$$

satisfies Condition 2.4.1 (GalRep- p).

Condition 2.4.3 (δ -reg). *There exists $\delta \in \mathbf{R}_{\geq 0}$ such that every cuspidal automorphic representation π whose archimedean component is a Δ^\vee -dominant and δ -regular discrete series satisfies Condition 2.4.1 (GalRep- p) for all $p \notin \text{Ram}(\mathbf{G})$.*

Remark 2.4.4. It is easy to see that Condition 2.4.3 implies Condition 2.4.2.

3. MAIN RESULTS

3.1. Group-theoretical Hasse invariants. Let p a prime number ($p = 2$ allowed). Let (G, μ) be a zip datum, and let P, Q be the attached parabolics with Levi subgroups respectively L and M , and let E be the attached zip group, as defined in §4.3. Denote by $G\text{-Zip}^\mu$ the stack of G -zips of type μ , as defined in Definition 4.3. By results of Moonen, Wedhorn, Pink and Ziegler, the stack $G\text{-Zip}^\mu$ is isomorphic to the quotient stack $[E \backslash G]$. There are finitely many E -orbits in G , parameterized by a subset ${}^I W$ of the Weyl group W of G (see §4.5). Let $C \subset G$ be an E -orbit, and let \overline{C} be its Zariski closure. We thus have a closed substack $[E \backslash \overline{C}]$. Identifying characters of L with characters of E through the natural maps $E \rightarrow P \rightarrow L$, we obtain for all $\chi \in \mathbf{X}^*(L)$ a line bundle $\mathcal{D}(\chi)$ on $G\text{-Zip}^\mu$. We fix a (compatible) Borel pair (B, T) defined over \mathbf{F}_p (see §4.4).

The following two notions are need for the statement of our result on group-theoretical Hasse invariants.

Definition 3.1.1. Let $\chi \in X^*(T)$.

- (1) We say that $\chi \in X^*(L)$ is ample if the induced line bundle on G/P_+ is ample, where P_+ is the parabolic opposite to P .
- (2) We say that χ is orbitally p -close if, for every coroot α^\vee satisfying $\langle \chi, \alpha^\vee \rangle \neq 0$ and all $\sigma \in W \rtimes \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we have:

$$\left| \frac{\langle \chi, \sigma \alpha^\vee \rangle}{\langle \chi, \alpha^\vee \rangle} \right| \leq p - 1.$$

Note that a quasi-constant character (Definition 2.1.1) is orbitally p -close for all primes p .

Theorem 3.1.2 (Group-theoretical Hasse invariants). *Let $C \subset G$ be an E -orbit and let \overline{C} denote its Zariski closure, with the reduced scheme structure. Let $\chi \in X^*(L)$ be an ample and orbitally p -close character of L . Then there exists an integer $N \geq 1$ and a section $h \in H^0([E \setminus \overline{C}], \mathcal{D}(\chi)^{\otimes N})$ whose non-vanishing locus is exactly the substack $[E \setminus C]$.*

We call h a group-theoretical Hasse invariant. To prove the existence of such sections, we introduce the stack of zip flags of type μ , denoted by $G\text{-ZipFlag}^\mu$. It has a natural projection $\pi : G\text{-ZipFlag}^\mu \rightarrow G\text{-Zip}^\mu$. The stack of zip flags is a group-theoretical generalization of the flag space studied by Ekedahl-Van de Geer for Siegel-type Shimura varieties in [EvdG09]. Furthermore, it carries a natural algebraic filtration, in the sense that the strata of $G\text{-ZipFlag}^\mu$ are defined as the fibers of a certain morphism of stacks $G\text{-ZipFlag}^\mu \rightarrow [(B^- \times B^-) \backslash G]$. After introducing this stack, the existence of Hasse invariants is a consequence of Chevalley's formula on the Schubert stack $[(B^- \times B^-) \backslash G]$.

We note that in general, both conditions "ample" and "orbitally p -close" on the character χ are necessary. However, when C is the open E -orbit of G , the condition "orbitally p -close" can be removed, thus recovering the result of [KW14] in the general Hodge-type case. Moreover, our Hasse invariants exist even if the zip datum (G, μ) does not arise as the zip datum of a Shimura variety. For example, we do not impose that the cocharacter μ is minuscule. We deduce from Theorem 3.1.2 the following corollary:

Corollary 3.1.3 (Hasse invariants). *Assume that (\mathbf{G}, \mathbf{X}) is a Shimura datum of Hodge type. Then for every Ekedahl-Oort stratum S in the mod p reduction $\text{Sh}_{\mathcal{K}}^1$, there exists $N \geq 1$ and a Hecke-equivariant section $h \in H^0(\overline{S}, \omega^{\otimes N})$ whose non-vanishing locus is exactly S .*

The line bundle ω already exists on the stack $G\text{-Zip}^\mu$, and corresponds to a certain character $\eta_\omega \in X^*(L)$.

3.2. Extension of Ekedahl-Oort strata. The Ekedahl-Oort stratification of the special fiber $\text{Sh}_{\mathcal{K}}^1$, which was introduced by Ekedahl-Oort in the Siegel case [Oor01] and generalized to the PEL case by Moonen [Moo01], has been extended to the Hodge case by Zhang [Zha] and Wortmann [Wor], using the Pink-Wedhorn-Ziegler theory of G -Zips [PWZ]. More precisely, Zhang and Wortmann construct a universal G -zip \mathcal{I} of type μ over the special fiber of a Hodge-type Shimura variety. This universal object induces a morphism of stacks:

$$(3.2.1) \quad \zeta : \text{Sh}_{\mathcal{K}}^1 \longrightarrow G\text{-Zip}^\mu$$

One defines the Ekedahl-Oort strata in $\mathrm{Sh}_{\mathcal{K}}^1$ as the fibers of this morphism. The points of the underlying topological space of $G\text{-Zip}^\mu$ are in bijection with the set ${}^I W$ by the parametrization (4.5.1). For $w \in {}^I W$, we will denote by S_w the corresponding stratum in $\mathrm{Sh}_{\mathcal{K}}^1$. The stratum S_e is zero-dimensional.

Zhang shows that the morphism ζ is smooth (Theorem 3.1.2 in [Zha]), generalizing part of a result of Viehmann-Wedhorn in the PEL-case (Theorem 5.1 in [VW13]). For example, one then deduces the dimensions of the strata. However, contrary to Viehmann-Wedhorn, Zhang does not prove the surjectivity of the map ζ . It was proved as a corollary of quasi-affineness of strata in [Kos14] that in the projective case, ζ is surjective. We extend the Ekedahl-Oort stratification to a toroidal compactification $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$, by proving the following lemma:

Lemma 3.2.1. *The G -zip \mathcal{I} extends to a G -zip $\mathcal{I}^{\mathrm{tor}}$ on $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$.*

The extended G -zip then yields a factorization:

$$\begin{array}{ccc} & \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1} & \\ \uparrow & \searrow \zeta^{\mathrm{tor}} & \\ \mathrm{Sh}_{\mathcal{K}}^1 & \xrightarrow{\zeta} & G\text{-Zip}^\mu \end{array}$$

Similarly, we denote by S_w^{tor} the stratum of $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ corresponding to $w \in {}^I W$. Motivated by the work of Ekedahl-van der Geer [EvdG09, §5] and Ekedahl-Oort [Oor01], we define the (extended) Ekedahl-Oort strata S_w^{min} in the minimal compactification $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{min},1}$ as the images of the S_w^{tor} by the projection $\pi_1^{\mathrm{tor},\mathrm{min}} : \mathrm{Sh}_{\mathcal{K}}^{\mathrm{min},1} \rightarrow \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$.

- Remark 3.2.2.* (1) In the general Hodge case Lan-Stroh have provided a general analysis of extending stratifications to toroidal and minimal compactifications [LS]. Stroh has informed us (personal communication) that the results of *loc. cit.* should imply that the morphism ζ^{tor} constructed in Lemma 3.2.1 is smooth, thereby extending Zhang's result from the open Shimura variety to its compactifications.
- (2) At this time, we do not know that the S_w^{tor} and the S_w^{min} satisfy the stratification property in general, however see (3) below. The stratification property for both toroidal and minimal strata would follow from the smoothness of ζ^{tor} .
- (3) In the Siegel case, Ekedahl-van der Geer show that the toroidal and minimal strata satisfy the stratification property [EvdG09, §5]. More generally, in the PEL case this has been proved by Boxer in [Box, Theorem B].

3.3. Affineness of Ekedahl-Oort Strata. In the Siegel case, Oort conjectured that the extended strata S_w^{min} are affine. This conjecture remained open until our work and that of Boxer.

Theorem 3.3.1 (Affineness). *Suppose (\mathbf{G}, \mathbf{X}) is a Shimura datum of Hodge type.*

- (1) *Assume that (\mathbf{G}, \mathbf{X}) is of compact type. Then all Ekedahl-Oort strata in the special fiber $\mathrm{Sh}_{\mathcal{K}}^1$ are affine.*
- (2) *Suppose that (\mathbf{G}, \mathbf{X}) is of noncompact type. Then the (extended) Ekedahl-Oort strata S_w^{min} are pairwise disjoint, locally closed and affine.*

Remark 3.3.2. In the PEL case, Theorem 3.3.1 has been proved independently by Boxer [Box, Theorem C]. Prior to our work and that of Boxer, to our knowledge the results known about affineness were the following: (i) Ito treated the Shimura varieties studied by Harris-Taylor [Ito05] (ii) Goldring-Nicole treated the μ -ordinary stratum for all PEL type A Shimura varieties [GN14] and (iii) Koskivirta-Wedhorn treated the μ -ordinary locus for a general Shimura variety of Hodge type [KW14]. In the Siegel case, Ekedahl-van der Geer proved a closely related result, that all flag space strata which do not intersect the boundary are affine for sufficiently large p [EvdG09, Lemma 10.5].

3.4. Factorization of coherent cohomology Hecke algebras. Our most fundamental application of group-theoretical Hasse invariants is the following result:

Theorem 3.4.1 (Reduction to H^0). *Suppose that $\varphi : (\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(2g), \mathbf{X}_g)$ is an embedding of Shimura data and $p \notin \text{Ram}(\mathbf{G})$. Let $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$. If (\mathbf{G}, \mathbf{X}) is neither of PEL type, nor of compact type, we make the following assumptions:*

- (1) *Suppose that either $R^i \pi_{1,*}^{\text{tor}, \min} \mathcal{V}_{\eta}^{\text{sub}} = 0$ for all $i > 0$, or that $p > \dim(\mathbf{G}, \mathbf{X})$ and $\eta \in \mathbf{Q}\eta_{\omega}$.*
- (2) *Assume $S_e = S_e^{\text{tor}}$.*

Then we conclude that, for every triple (i, n, \mathcal{K}) consisting of $i \in \mathbf{Z}_{\geq 0}$, $n \in \mathbf{Z}_{\geq 1}$, and a p -hyperspecial level $\mathcal{K} \subset \mathbf{G}(\mathbf{A}_f)$, there exists $a', b \in \mathbf{Z}_{\geq 1}$, such that, for all $a \in \mathbf{Z}_{\geq 1}$ with $a \equiv b \pmod{a'}$, one has the following commutative triangle of Hecke algebras:

$$(3.4.1) \quad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{H}_{\mathcal{K}}(0, n, \eta + a\eta_{\omega}) \\ & \searrow & \downarrow \\ & & \mathcal{H}_{\mathcal{K}}(i, n, \eta) \end{array}$$

We define in §10.3.1 a generalization of the flag space of Ekedahl-Van der Geer introduced in [EvdG09], whose special fiber is the fiber product of $\text{Sh}_{\mathcal{K}}^1$ and $G\text{-ZipFlag}^{\mu}$ over $G\text{-Zip}^{\mu}$. An argument using analogues of Hasse invariants on the flag space gives the following strengthening of Theorem 3.4.1:

Corollary 3.4.2. *Keep the assumptions of Theorem 3.4.1. Let $\chi \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ be in the image of $X_{+}^*(\mathbf{T}_{\mathbf{C}})$ under the map $h_{w_0, M}$ (defined in (6.1.3)). Then, for every a satisfying (3.4.1), one has the additional commutative triangle:*

$$(3.4.2) \quad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{H}_{\mathcal{K}}(0, n, \eta + a\eta_{\omega} + \chi) \\ & \searrow & \downarrow \\ & & \mathcal{H}_{\mathcal{K}}(i, n, \eta) \end{array}$$

In the case $n = 1$ (the special fiber), one deduces immediately a more elementary statement in terms of systems of Hecke eigenvalues. Suppose κ is a finite field, and M is a finite dimensional κ -vector space which is also an \mathcal{H} -module. We say that a system of Hecke eigenvalues $(b_T)_{T \in \mathcal{H}}$ appears in M if there exists $m \in M$ and a finite extension κ'/κ such that $Tm = b_T m$ in $M \otimes \kappa'$ for all $T \in \mathcal{H}$.

Corollary 3.4.3 (Reduction to H^0 , modulo p). *Let a be as in Theorem 3.4.1. Then every system of Hecke eigenvalues $\{b_T\}_{T \in \mathcal{H}}$ which appears in $H^i(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}, \mathcal{V}_{\eta}^{\mathrm{sub}})$ also appears in $H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}, \mathcal{V}_{\eta+a\eta_{\omega}}^{\mathrm{sub}})$ and $H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}, \mathcal{V}_{\eta+a\eta_{\omega}+\chi}^{\mathrm{sub}})$.*

Remark 3.4.4 (Concerning our assumptions). We expect it will soon be shown that assumptions (1)-(2) of Theorem 3.4.1 always hold. These assumptions hold in the PEL case, which is why we don't need to assume them in that case. In the PEL case, the vanishing of higher direct images as in assumption (1) is a theorem of Lan [Lan, Theorem 8.2.1.3] (see also [LS14] for a simpler argument under a mild additional hypothesis). In the Hodge case, when $\eta \in \mathbf{Q}\eta_{\omega}$, the vanishing is a theorem of Stroth [Str]. We explain in Lemma 9.3.1 why it intersects the boundary trivially.

3.5. Irregular Galois representations and pseudo-representations. We have two basic results concerning the association of Galois representations (resp. pseudo-representations) to automorphic representations whose archimedean component is a non-degenerate limit of discrete series (resp. torsion in coherent cohomology). Our first result, Theorem 3.5.1, states that, under very mild hypotheses on (\mathbf{G}, \mathbf{X}) , every automorphic representation π of \mathbf{G} whose archimedean component π_{∞} is a non-degenerate limit of discrete series satisfies Condition 2.4.1 (GalRep- p) away from $\mathrm{Ram}(\mathbf{G}, \pi)$. Moreover, each of the Hecke algebras $\mathcal{H}_{\mathcal{K}}(i, n, \eta)$ receives a pseudo-representation. Second, using known results special to $GL(m)$ on the existence of regular automorphic Galois representations, Theorems 3.5.5 and 3.5.6 give analogous and unconditional results for arbitrary unitary similitude groups.

We hope that by carefully stating our hypotheses in a general context, it will be easy to apply Theorem 3.5.1 as soon as new results are established for cohomological representations. For example, it is possible that Theorem 3.5.1 can be combined with the work in progress of Kret-Shin on $GSpin(2m+1)$ -valued Galois representations associated to automorphic representations of symplectic similitude groups.

3.5.1. *The general result.*

Theorem 3.5.1. *Suppose that $(\mathbf{G}, \mathbf{X}, r)$ is a triple satisfying Condition 2.4.2 (ERG- p), where (\mathbf{G}, \mathbf{X}) is a Hodge-type Shimura variety and $r : {}^L\mathbf{G} \rightarrow GL(m)$. Let $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ and $p \notin \mathrm{Ram}(\mathbf{G})$. Assume (\mathbf{G}, \mathbf{X}) satisfies assumptions (1)-(2) of Theorem 3.4.1. Then:*

[LDS] *If π is a cuspidal automorphic representation of \mathbf{G} with*

$$\pi_{\infty} = \pi(-w_{0,c}(\eta + \rho), \mathcal{C}) \text{ for some } \mathcal{C},$$

then the pair (π, r) satisfies Condition 2.4.1 (GalRep- p) for all $p \notin \mathrm{Ram}(\pi)$.

[Tor] *For every quadruple $(i, n, \eta, \mathcal{K})$ with $i \in \mathbf{Z}_{\geq 0}$, $n \in \mathbf{Z}_{\geq 1}$, $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ and a p -hyperspecial level \mathcal{K} , there is a unique pseudo-representation*

$$(3.5.1) \quad R_{p,i}(r; i, n, \eta, \mathcal{K}) : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathcal{H}_{\mathcal{K}}(i, n, \eta),$$

satisfying, for all $j \in \mathbf{Z}_{\geq 1}$,

$$(3.5.2) \quad R_{p,i}(r; i, n, \eta, \mathcal{K})(\mathrm{Frob}_v^j) = T_v^{(j)}(r; i, n, \eta, \mathcal{K}).$$

We note the special case when $(\mathbf{G}, \mathbf{X}) = (GSp(2g), \mathbf{X}_g)$. Since $(GSp(2g), \mathbf{X}_g)$ is of PEL type and has $\mathrm{Ram}(\mathbf{G}) = \emptyset$, part [LDS] of the theorem implies:

Corollary 3.5.2. *Suppose that Condition 2.4.2 (ERG- p) holds for a representation r of ${}^L GSp(2g)$. Then for every cuspidal automorphic representation π of $GSp(2g)$ with π_∞ a C -algebraic, non-degenerate limit of discrete series, the pair (π, r) satisfies Condition 2.4.1 (GalRep- p) for all primes p . Moreover, conclusion [Tor] of Theorem 3.5.1 holds.*

Specializing even further to $GSp(4)$, Condition 2.4.3 is known with $\delta = 0$ when r is the standard four-dimensional representation. This can be deduced in two ways: By using the work of Arthur [Art04] to transfer to $GL(4)$ and then applying Shin [Shi11], or by the work of Weissauer [Wei05] and Laumon [Lau97], [Lau05]. Thus we have the following unconditional result:

Corollary 3.5.3. *Let r be the standard four-dimensional representation of $GSp(4)$. For every cuspidal automorphic representation π of $GSp(4)$ with π_∞ a C -algebraic, non-degenerate limit of discrete series, the pair (π, r) satisfies Condition 2.4.1 (GalRep- p) for all primes p . Moreover, conclusion [Tor] of Theorem 3.5.1 holds.*

Remark 3.5.4 (Variants). Upon consulting the proof of Theorem 3.5.1, it should be clear to the reader that the proof can easily be adapted to treat several variants of the theorem. One notable variant is to modify Condition 2.4.2 with an analogous one where $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is replaced by $\text{Gal}(\overline{F}/F)$ for some number field F which plays a special role for the Shimura variety and then to do the same in the statement of the theorem. Our results for unitary groups in §3.5.2 already give an example of this. In the Hilbert case, taking F to be the totally real field, we recover the main result of [ERX].

3.5.2. Unitary Similitude Groups. Our first result about unitary similitude groups is a direct generalization of the first author’s previous work [Gol14, Th. 1.2.1] and his joint work with Marc-Hubert Nicole [GN14, Th. 1.4]. We could state our result simply by saying that one can replace “ \mathbf{X} -holomorphic” in [GN14, Th. 1.4] with “non-degenerate” at the expense of assuming that π_p is unramified. Our second result concerns torsion and produces Hecke algebra-valued pseudo-representations.

We also take this opportunity to improve the local-global compatibility statement to one that includes all but finitely many primes, whereas the above mentioned results were only for a density one set of primes. For this we have found useful the description of local base change from unitary similitude groups to $GL(m)$ at unramified primes in [HLTT13, §1.3]¹⁰

In the terminology of [Gol14, §3.1], suppose $\mathcal{U} = (B, V, *, <, >, \tilde{h})$ is a unitary Kottwitz datum with associated Shimura datum (\mathbf{G}, \mathbf{X}) . Let F be the imaginary CM field given by the center of B and let F^+ denote its maximal totally real subfield. Put $m = (\dim_F \text{End}_B V)^{1/2}$.

Let π be a cuspidal automorphic representation of \mathbf{G} . For every $v \notin \text{Ram}(\mathbf{G}, \pi)$ and w a place of F above v , define the base change of π_v , denoted $\text{BC}(\pi_v)$, and its w -part $\text{BC}(\pi_v)_w$ as in [HLTT13, §1.3].

Theorem 3.5.5 (LDS, unitary case). *Suppose π is a cuspidal automorphic representation of \mathbf{G} whose archimedean component π_∞ is a C -algebraic, non-degenerate limit of discrete series. Assume $p \notin \text{Ram}(\mathbf{G}, \pi)$. Then there exists a unique*

¹⁰This seems to be one of many things that has been potentially “well-known to experts” for along time, but difficult to extract from the literature prior to [HLTT13].

semisimple Galois representation

$$(3.5.3) \quad R_{p,\iota}(\pi) : \text{Gal}(\overline{F}/F) \longrightarrow GL(m, \overline{\mathbf{Q}}_p)$$

such that, for all primes w of F which lie over some prime $v \notin \text{Ram}(\mathbf{G}, \pi)$, the representation $R_{p,\iota}(\pi)$ is unramified at w and there is an isomorphism of Weil-Deligne representations

$$(3.5.4) \quad (R_{p,\iota}(\pi)|_{W_{F_w}})^{\text{ss}} \cong \iota^{-1} \text{rec}_{F_w} \left(\text{BC}(\pi_v)_w \otimes |\cdot|_w^{\frac{1-m}{2}} \right),$$

where W_{F_w} is the Weil group of F_w , the superscript ss denotes semi-simplification and rec_{F_w} is the local Langlands correspondence, normalized as in [HT01].

Given $v \notin \text{Ram}(\mathbf{G}) \cup \{p, \infty\}$ and $j \in \mathbf{Z}_{\geq 1}$, let $T_v^{(j),U}$ be the Hecke operator defined between Lemma 6.1 and Lemma 6.2 of [HLTT13] and denoted $T_v^{(i)}$ there. Let $T_v^{(j),U}(i, n, \eta, \mathcal{K})$ be the image of $T_v^{(j),U}$ in $\mathcal{H}_{\mathcal{K}}(i, n, \eta)$.

Concerning torsion, we show:

Theorem 3.5.6 (Torsion, unitary case). *Assume $p \notin \text{Ram}(\mathbf{G})$ and let \mathcal{K} be a p -hyperspecial level. Then for every quadruple $(i, n, \eta, \mathcal{K})$, with $i \in \mathbf{Z}_{\geq 0}$, $n \in \mathbf{Z}_{\geq 1}$ and $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{G}})$ there exists a unique pseudo-representation*

$$(3.5.5) \quad R_{p,\iota}(i, n, \eta, \mathcal{K}) : \text{Gal}(\overline{F}/F) \rightarrow \mathcal{H}_{\mathcal{K}}(i, n, \eta),$$

satisfying, for all $j \in \mathbf{Z}_{\geq 1}$ and all $v \in \text{Ram}(\mathbf{G}) \cup \{p, \infty\}$,

$$(3.5.6) \quad R_{p,\iota}(i, n, \eta, \mathcal{K})(\text{Frob}_v^j) = T_v^{(j),U}(i, n, \eta, \mathcal{K}).$$

3.6. Serre's letter to Tate for Hodge type Shimura varieties. As a by-product of our methods, we are able to generalize (part of) the generalizations by Ghitza [Ghi04] and Reduzzi [Red03] of Serre's letter to Tate [Ser96].

In *loc. cit.*, Serre showed that the systems of Hecke eigenvalues (for \mathcal{H}) appearing in

$$(3.6.1) \quad \bigoplus_{k=1}^{\infty} H^0(\text{Sh}_{1,\mathcal{K}}^1, \omega^k)$$

are the same as those which appear in

$$(3.6.2) \quad \bigoplus_{k=1}^{\infty} H^0(S_e, \omega^k).$$

In this particular situation, we note that the complicated notation means simply that $\text{Sh}_{1,\mathcal{K}}^1$ is the special fiber of the modular curve of level \mathcal{K} and S_e is its supersingular locus. Serre then went on to show that these systems of Hecke eigenvalues are also precisely the ones that appear for –what later became known as– algebraic modular forms in the sense of Gross for the inner form of $GL(2)$ deduced from the quaternion algebra ramified exactly at $\{p, \infty\}$.

Both of Serre's results were generalized to the Siegel case by Ghitza. They were further generalized by Reduzzi to a very restricted class of PEL-type Shimura varieties. Namely, Reduzzi only considers (\mathbf{G}, \mathbf{X}) such that \mathbf{G} is a unitary similitude group associated to an imaginary quadratic field¹¹ (not a general CM field) such

¹¹Reduzzi claims his result generalizes to all Shimura varieties of PEL-type (A) or (C) whose classical superspecial locus is nonempty, but we could not find neither a precise statement, nor a proof of this generalization in his paper.

that the classical superspecial locus of $\mathrm{Sh}_{\mathcal{K}}^1$ is nonempty¹². In this case, the stratum S_e coincides with the classical superspecial locus.

This condition is unnatural: It is easy to see that, similar to the distinction between the classical ordinary locus and the μ -ordinary locus, examples of PEL type abound where the classical superspecial locus is empty. For example, if (\mathbf{G}, \mathbf{X}) is of unitary type, p splits completely in E and $E \neq \mathbf{Q}$, then the classical superspecial locus of $\mathrm{Sh}_{\mathcal{K}}^1$ is empty.

The following theorem provides a generalization of the first of Serre's equivalences to Hodge-type Shimura varieties, by replacing the classical superspecial locus with the stratum S_e .

Theorem 3.6.1. *Let (\mathbf{G}, \mathbf{X}) be of Hodge type. Then the systems of Hecke eigenvalues appearing in*

$$(3.6.3) \quad \bigoplus_{\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})} H^0(\mathrm{Sh}_{\mathcal{K}}^1, \mathcal{V}_{\eta})$$

are the same as those which appear in

$$(3.6.4) \quad \bigoplus_{\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})} H^0(S_e, \mathcal{V}_{\eta}).$$

In particular the number of such systems of Hecke eigenvalues is finite.

Combining Cor. 3.4.3 with Th. 3.6.1, one obtains immediately:

Corollary 3.6.2. *Make the assumptions of Theorem 3.4.1. Then the systems of Hecke eigenvalues appearing in*

$$(3.6.5) \quad \bigoplus_{i \in \mathbf{Z}_{\geq 0}, \eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})} H^i(\mathrm{Sh}_{\mathcal{K}}^1, \mathcal{V}_{\eta})$$

also appear in (3.6.4). In particular the number of systems appearing in (3.6.5) is finite.

We hope to return to the second of Serre's equivalences in our level of generality in future work.

4. THE STACK OF G -ZIPS

Let k denote an algebraic of \mathbf{F}_p . All schemes are schemes of finite type over k . We will consider only left actions of algebraic groups on schemes.

4.1. Preliminaries on quotient stacks.

¹²To avoid confusion, we specify that for us the classical superspecial locus means those points whose underlying abelian scheme is isomorphic to a product of supersingular elliptic curves. Some people call this locus simply the "superspecial locus", while others reserve the term "superspecial locus" for the zero-dimensional Ekedahl-Oort stratum. The latter two conventions are jointly incompatible.

4.1.1. *G-varieties and quotient stacks.* If an algebraic group G acts on a scheme X , we denote by $\text{Pic}^G(X)$ the group of G -linearized line bundles on X . The image of the natural map $\text{Pic}^G(X) \rightarrow \text{Pic}(X)$ consists of G -linearizable line bundles. We have a natural identification $\text{Pic}^G(X) \simeq \text{Pic}([G \backslash X])$. We denote by $\lambda \mapsto \mathcal{L}(\lambda)$ the natural map $X^*(G) \rightarrow \text{Pic}^G(X)$. For any line bundle \mathcal{L} on $[G \backslash X]$, we have an identification

$$(4.1.1) \quad H^0([G \backslash X], \mathcal{L}) \simeq H^0(X, \pi^* \mathcal{L})^G$$

where $\pi : X \rightarrow [G \backslash X]$ is the natural projection. In particular, for $\lambda \in X^*(G)$ one has:

$$(4.1.2) \quad H^0([G \backslash X], \mathcal{L}(\lambda)) = \{f : X \rightarrow k, f(g \cdot x) = \lambda(g)f(x), \forall g \in G, x \in X\}.$$

Lemma 4.1.1. *Let G be an algebraic group acting on an integral scheme X containing an open G -orbit U . For any line bundle \mathcal{L} on the stack $[G \backslash X]$, the k -vector space $H^0([G \backslash X], \mathcal{L})$ has dimension less than 1.*

Proof. See [KW14, Prop.1.18]. □

4.2. **Morphisms of quotient stacks.** Let $f : X \rightarrow Y$ be a morphism of schemes and let G, H be group schemes such that G acts on X and H acts on Y . Assume that there is a morphism of schemes: $\alpha : G \times X \rightarrow H$ such that the following diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{(\alpha, f)} & H \times Y \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

The vertical maps are the natural maps given by the group actions. The commutativity of this diagram is equivalent to the following identity:

$$(4.2.1) \quad f(g \cdot x) = \alpha(g, x) \cdot f(x), \quad g \in G, x \in X.$$

Assume further that α satisfies the cocycle condition:

$$(4.2.2) \quad \alpha(gg', x) = \alpha(g, g' \cdot x) \alpha(g', x) \quad g, g' \in G, x \in X.$$

Then α and f induce a morphism between the groupoids attached to (G, X) and (H, Y) . This morphism of groupoids induces in turn a map of stacks $\tilde{f} : [G \backslash X] \rightarrow [H \backslash Y]$, such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ [G \backslash X] & \xrightarrow{\tilde{f}} & [H \backslash Y] \end{array}$$

where the vertical maps are the obvious projections. On topological spaces, the map \tilde{f} sends the G -orbit of x to the H -orbit of $f(x)$. Note that this map is well-defined by equation (4.2.1).

4.3. Definition of $G\text{-Zip}^\mu$ in terms of torsors. We denote by k an algebraic closure of \mathbf{F}_p . Let G be a connected reductive group over k , defined over \mathbf{F}_p . Denote by $\varphi : G \rightarrow G$ the relative Frobenius. Let $\mu : \mathbf{G}_{m,k} \rightarrow G$ be a cocharacter, not necessarily defined over \mathbf{F}_p . We will call the pair (G, μ) a *zip datum*. The cocharacter μ gives rise to a pair of opposite parabolics (P_-, P_+) and a Levi subgroup $L := P_- \cap P_+$. The Lie algebra of the parabolic P_- (resp. P_+) is the sum of the non-positive (resp. non-negative) weight spaces of $\text{Ad} \circ \mu$. We set $P := P_-$, $Q := (P_+)^{(p)}$ and $M := L^{(p)}$. Then M is a Levi subgroup of Q . We denote by $R_u P$ and $R_u Q$ the unipotent radicals of P and Q , respectively. Pink, Wedhorn, Ziegler ([PWZ], Definition 1.4) give the following definition:

Definition 4.3.1. A G -zip of type μ over a k -scheme S is a tuple $\underline{I} = (I, I_P, I_Q, \iota)$ where I is a G -torsor over S , $I_P \subset I$ is a P -torsor over S , $I_Q \subset I$ is a Q -torsor over S , and $\iota : (I_P)^{(p)}/R_u(P)^{(p)} \rightarrow I_Q/R_u(Q)$ an isomorphism of M -torsors.

The category of G -zips over S is denoted by $G\text{-Zip}^\mu(S)$. The $G\text{-Zip}^\mu(S)$ give rise to a fibered category $G\text{-Zip}^\mu$ over the category of k -schemes, which is a smooth stack of dimension 0. The Frobenius restricts to a map $\varphi : L \rightarrow M$. The inclusions $L \subset P$ and $M \subset Q$ induce isomorphisms $L \simeq P/R_u P$ and $M \simeq Q/R_u Q$. We get natural maps $P \rightarrow L$ and $Q \rightarrow M$ which we denote by $x \mapsto \bar{x}$. We define the zip group E as:

$$E := \{(a, b) \in P \times Q, \varphi(\bar{a}) = \bar{b}\}.$$

The group $G \times G$ acts on G by the rule $(a, b) \cdot g := agb^{-1}$. By restriction, the groups $P \times Q$ and E also act on G . By Theorem 1.5 in *loc. cit.*, there is an isomorphism of stacks:

$$G\text{-Zip}^\mu \simeq [E \backslash G].$$

4.4. Zip data of Hodge type. The group $G(k)$ acts on the group of cocharacters by conjugation. Let (G, μ) and (G, μ') be two zip data such that $\mu' = g \cdot \mu$ for some $g \in G(k)$. Denote by P, Q, E and P', Q', E' the corresponding groups. Then one has

$$P' = gPg^{-1} \quad \text{and} \quad Q' = \varphi(g)Q\varphi(g)^{-1}.$$

Furthermore, the maps

$$\alpha : E \rightarrow E', \quad (a, b) \mapsto (gag^{-1}, \varphi(g)b\varphi(g)^{-1})$$

$$f : G \rightarrow G, \quad x \mapsto gx\varphi(g)^{-1}$$

satisfy the conditions of section 4.2 and induce an isomorphism of quotient stacks:

$$[E \backslash G] \simeq [E' \backslash G].$$

Definition 4.4.1. Let (G_1, μ_1) and (G_2, μ_2) be two zip data. A morphism of zip data $f : (G_1, \mu_1) \rightarrow (G_2, \mu_2)$ is a morphism of groups $f : G_1 \rightarrow G_2$ defined over \mathbf{F}_p , such that $\mu_2 = f \circ \mu_1$. We say that f is an embedding of zip data if f is injective.

We obtain in this way the category of zip data. If $f : (G_1, \mu_1) \rightarrow (G_2, \mu_2)$ is a morphism of zip data, it induces naturally a morphism of stacks $[E_1 \backslash G_1] \rightarrow [E_2 \backslash G_2]$.

Let (B, T) be a Borel pair, and let B^- be the opposite Borel to B with respect to T . We say that the pair (B, T) is μ -compatible if the following conditions are satisfied:

- (i) B, T are defined over \mathbf{F}_p .

(ii) One has $B^- \subset P$ and $B \subset Q$.

Note that in particular, the group G is quasi-split over \mathbf{F}_p if such a pair exists. One has the following lemma:

Lemma 4.4.2. *Let (G, μ) be a zip datum with G quasi-split over \mathbf{F}_p . Then there exists a cocharacter μ' conjugate to μ , such that the zip datum (G, μ') admits a compatible Borel pair.*

Proof. This is [KW14, Lemma 4.2]. \square

We now define zip data of Hodge-type. Let (V, ψ) be a symplectic space over \mathbf{F}_p , and let $GS\!p(V, \psi)$ be the symplectic group of (V, ψ) . Consider a decomposition $\mathcal{D} : V = W_+ \oplus W_-$ where W_+ and W_- are maximal isotropic subspaces, not necessarily defined over \mathbf{F}_p . Define a cocharacter $\mu_{\mathcal{D}} : \mathbf{G}_{m,k} \rightarrow GS\!p(V, \psi)$ such that an element $x \in \mathbf{G}_{m,k}$ acts trivially on W_- and acts by multiplication by x on W_+ . The groups attached to the zip datum $(GS\!p(V, \psi), \mu_{\mathcal{D}})$ are then:

$$P_{\mathcal{D}} := \text{Stab}_{GS\!p(V, \psi)}(W^+), \quad P_{\mathcal{D}}^- := \text{Stab}_{GS\!p(V, \psi)}(W_-), \quad Q_{\mathcal{D}} := (P_{\mathcal{D}}^-)^{(p)}.$$

The intersection $L_{\mathcal{D}} := P_{\mathcal{D}} \cap P_{\mathcal{D}}^-$ is a Levi subgroup of $P_{\mathcal{D}}$ and $M_{\mathcal{D}} := (L_{\mathcal{D}})^{(p)}$ is a Levi subgroup of $Q_{\mathcal{D}}$. Since the group $GS\!p(V, \psi)$ acts transitively on the set of all decompositions \mathcal{D} of V into maximal isotropic subspaces, all zip data obtained in this way are isomorphic. We call such a zip datum $(GS\!p(V, \psi), \mu_{\mathcal{D}})$ a Siegel-type zip datum.

Definition 4.4.3. *We say that the zip datum (G, μ) is of Hodge-type if there exists a Siegel-type zip datum $(GS\!p(V, \psi), \mu_{\mathcal{D}})$ and an embedding of zip data $\iota : (G, \mu) \rightarrow (GS\!p(V, \psi), \mu_{\mathcal{D}})$.*

4.5. Stratification. Let (G, μ) be a zip datum, and let P, Q, L, M, E be the attached groups, as above. The stack $G\text{-Zip}^{\mu}$ carries a natural stratification, corresponding to the E -orbits in G , which we now describe. We fix a compatible Borel pair (B, T) . Denote by $W := W(G, T)$ the Weyl group. Since T is defined over \mathbf{F}_p , we have an action of $\text{Gal}(k/\mathbf{F}_p)$ on W . We denote by $\varphi : W \rightarrow W$ the action of the geometric Frobenius on W . Let $\Phi \subset X^*(T)$ be the set of T -roots, Φ_+ the set of positive roots with respect to B (i.e. those appearing in $\text{Lie } B$), and let $\Delta \subset \Phi_+$ be the set of positive simple roots. For $\alpha \in \Delta$, let $s_{\alpha} \in W$ be the corresponding reflection. Then $(W, \{s_{\alpha}, \alpha \in \Delta\})$ is a Coxeter group. Let $\ell : W \rightarrow \mathbf{N}$ be the length function.

Recall that parabolic subgroups containing B correspond to subsets of Δ , such that B itself corresponds to the empty set and G to Δ . This induces a bijection between conjugation classes of parabolics in G and the powerset $\mathcal{P}(\Delta)$, which we normalize as follows: If P' is a parabolic of G containing B with Levi subgroup L' , the type of P' is defined to be $\Delta \cap \Phi(T, L')$. Denote by I the type¹³ of P .

Let $W_I \subset W$ be the subgroup generated by the s_{α} for $\alpha \in I$. Note that the Weyl group of L is then $w_0 W_I w_0$, because $B^- \subset P$. Define ${}^I W$ (resp. W^I) to be the subset of elements $w \in W$ which are minimal in the coset $W_I w$ (resp. $w W_I$). The set ${}^I W$ (resp. W^I) is a set of representatives for the quotients $W_I \backslash W$ (resp. W/W_I). For $w \in W$, we choose a representative $\dot{w} \in N_G(T)$, such that $(w_1 w_2)^{\cdot} = \dot{w}_1 \dot{w}_2$

¹³Beware that by definition P contains B^- and not B

whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$. Let w_0 (resp. $w_{0,I}$) be the longest element in W (resp. W_I), and define $w_1 = w_0 w_{0,I}$. Then the map

$$(4.5.1) \quad {}^I W \rightarrow \{E\text{-orbits in } G\}, \quad w \mapsto E \cdot (\dot{w}_1 \dot{w})$$

defines a bijection between the set ${}^I W$ and the set of E -orbits in G (see Theorem 7.5 in [PWZ11]). There is a dual parametrization (§11 in *loc. cit.*) given by a bijection:

$$(4.5.2) \quad W^I \rightarrow \{E\text{-orbits in } G\}, \quad w \mapsto E \cdot (\dot{w}_1 \dot{w}).$$

5. THE STACK OF ZIP FLAGS

5.1. Definition of the stack of zip flags. We fix a zip datum (G, μ) , and let $G\text{-Zip}^\mu$ be the stack of G -zips of type μ . By Lemma 4.4.2, we may assume without loss of generality that (G, μ) admits a μ -compatible Borel pair. We will fix throughout a μ -compatible Borel pair (B, T) .

Lemma 5.1.1. *The μ -compatible Borel pairs have the form (gBg^{-1}, gTg^{-1}) for some $g \in P \cap Q \cap G(\mathbf{F}_p)T$.*

Proof. Let (B', T') be a compatible Borel pair. Since k is algebraically closed, there exists $g \in G$ such that $(B', T') = (gBg^{-1}, gTg^{-1})$. The conditions $gBg^{-1} \subset Q$ and $gB^-g^{-1} \subset P$ impose $g \in P \cap Q$. Since (B', T') is defined over \mathbf{F}_p , one has $\varphi(g)^{-1}g \in N_G(B) = B$. But the opposite Borel gB^-g^{-1} is also defined over \mathbf{F}_p , so similarly $\varphi(g)^{-1}g \in B \cap B^- = T$. The statement follows readily by Lang-Steinberg's theorem. \square

Definition 5.1.2. *A G -zip flag of type (μ, B, T) over a k -scheme S is a pair $\hat{I} = (\underline{I}, J)$ where $\underline{I} = (I, I_P, I_Q, \iota)$ is a G -zip over S , and $J \subset I_P$ is a B^- -torsor.*

We denote by $G\text{-ZipFlag}^{(\mu, B, T)}(S)$ the category of G -zip flags over S . By the same arguments as for G -zips, we obtain a stack $G\text{-ZipFlag}^{(\mu, B, T)}$ over k , which we call the stack of G -zip flags of type (μ, B, T) . One sees immediately that this stack is (up to isomorphism) independant of the choice of a μ -compatible pair (B, T) . Hence we will simply denote it by $G\text{-ZipFlag}^\mu$. There is a natural morphism of stacks

$$\pi : G\text{-ZipFlag}^\mu \longrightarrow G\text{-Zip}^\mu$$

which forgets the B^- -torsor J on the level of categories $G\text{-ZipFlag}^\mu(S) \rightarrow G\text{-Zip}^\mu(S)$. We now describe this stack as a quotient stack.

We consider the action of $E \times B^-$ on $G \times P$ defined as follows: For $(a, b) \in E$, $c \in B^-$ and $(g, r) \in G \times P$, we set:

$$((a, b), c) \cdot (g, r) := (agb^{-1}, arc^{-1}).$$

Let $pr_1 : G \times L \rightarrow G$ be the projection on the first factor. If $\alpha : E \times B^- \rightarrow E$ denotes the projection on E , one sees immediately that the pair (pr_1, α) induces a map of stacks $\pi : [(E \times B^-) \backslash (G \times P)] \rightarrow [E \backslash G]$, as explained in section §4.2.

Theorem 5.1.3. *There is a commutative diagram*

$$\begin{array}{ccc}
G\text{-ZipFlag}^\mu & \xrightarrow{\pi} & G\text{-Zip}^\mu \\
\downarrow \simeq & & \downarrow \simeq \\
[(E \times B^-) \backslash (G \times P)] & \xrightarrow{pr_1} & [E \backslash G]
\end{array}$$

The vertical maps are isomorphisms and the lower horizontal map is induced by the projection on the first factor.

Proof. The proof is similar to the proof of Proposition 3.11 in [PWZ11]. Let S be a scheme over k . To $(g, r) \in G \times P(S)$, we will attach a G -zip flag $\hat{I}_{(g,r)} = (\underline{I}_{(g,r)}, J_{(g,r)})$ over S . Let $\underline{I}_{(g,r)} := \underline{I}_g$ be the G -zip over S attached to g by Construction 3.4 in *loc. cit.* Define $J_{(g,r)}$ as the image of $B^- \times S \subset P \times S$ under left multiplication by r . It follows from Lemma 3.5 in *loc. cit.* that every G -zip flag is étale-locally isomorphic to $\hat{I}_{(g,r)}$ for some $(g, r) \in G \times P(S)$.

For two elements $(g, r), (g', r') \in G \times P(S)$ We define the transporter as

$$(5.1.1) \quad \text{Transp}((g, r), (g', r')) := \{(\epsilon, b) \in E \times B^-(S) \mid (\epsilon, b) \cdot (g, r) = (g', r')\}.$$

We claim that there is a natural bijection between $\text{Transp}((g, r), (g', r'))$ and the set of morphisms $\phi : \hat{I}_{(g,r)} \rightarrow \hat{I}_{(g',r')}$. A morphism $\varphi : \hat{I}_{(g,r)} \rightarrow \hat{I}_{(g',r')}$ consists in particular of a morphism $\underline{I}_g \rightarrow \underline{I}_{g'}$. By Lemma 3.10 of *loc. cit.*, one can attach to it an element $\epsilon = (p_+, p_-) \in E(S)$, using the same notation. By compatibility, the map $J_{(g,r)} \rightarrow J_{(g',r')}$ must be induced by left multiplication by p_+ . Since $J_{(g,r)} = r(S \times B^-)$ and $J_{(g',r')} = r'(S \times B^-)$, one has the condition $b := r'^{-1}p_+r \in B^-(S)$. The map $\phi \mapsto (\epsilon, b)$ defines then a bijection. This terminates the proof of the claim. The isomorphism of the diagram follows as in the proof of Proposition 3.11 of *loc. cit.* From our construction, it is clear that the diagram commutes. \square

Define $B_L := B \cap L$ and $B_L^- := B^- \cap L$. The quotient stack $[(E \times B^-) \backslash (G \times P)]$ is clearly isomorphic to $[(E \times B_L^-) \backslash (G \times L)]$, where the action is given by

$$((a, b), c) \cdot (g, r) := (agb^{-1}, \bar{a}rc^{-1})$$

for $(a, b) \in E$, $c \in B_L^-$, $(g, r) \in G \times P$, and $\bar{a} \in L$ the image of a by the natural map $P \rightarrow L$. Furthermore, there is a natural isomorphism:

$$(5.1.2) \quad [(E \times B_L^-) \backslash (G \times L)] \simeq [E \backslash (G \times (B_L^- \backslash L))].$$

5.2. The Schubert stack. We define the Schubert stack as the quotient stack $[(B^- \times B^-) \backslash G]$. For $w \in W$, we set

$$C_w := B^- \dot{w} B^-.$$

It is a locally closed subset of G of dimension $\dim(B) + \ell(w)$. We denote by $\overline{C_w}$ the Zariski closure of C_w in G , which we endow with the reduced structure. The following theorem is well-known ([RR85], Theorem 3):

Theorem 5.2.1. *For all $w \in W$, the variety $\overline{C_w}$ is normal.*

We identify the points of the underlying topological space of $[(B^- \times B^-) \backslash G]$ with the elements of W through the parametrization $w \mapsto B^- \dot{w} B^-$. For each $w \in W$, we have substacks $[(B^- \times B^-) \backslash C_w]$ and $[(B^- \times B^-) \backslash \overline{C_w}]$. For each pair of characters $(\lambda, \mu) \in X^*(T) \times X^*(T)$, we have a line bundle $\mathcal{L}_{(\lambda, \mu)}$ on $[(B^- \times B^-) \backslash G]$.

Theorem 5.2.2. *Let w be an element in W . One has the following:*

- (1) *The line bundle $\mathcal{L}_{(\lambda, \mu)}$ admits global sections on the stack $[(B^- \times B^-) \setminus C_w]$ if and only if the relation $\mu = w^{-1}\lambda$ holds.*
- (2) *The space $H^0([(B^- \times B^-) \setminus C_w], \mathcal{L}_{(\lambda, w^{-1}\lambda)})$ has dimension one.*
- (3) *The Weil divisor of any nonzero section $f \in H^0([(B^- \times B^-) \setminus C_w], \mathcal{L}_{(\lambda, w^{-1}\lambda)})$ (viewed as a rational function of $\overline{C_w}$) is equal to:*

$$\operatorname{div}(f) = \sum_{\alpha \in E_w} \langle \lambda, ww_0\alpha^\vee \rangle X_{ww_0s_\alpha w_0}$$

where E_w is the set of roots $\alpha \in \Phi$ such that $ww_0s_\alpha w_0 < w$ with length equal to $\ell(w) - 1$.

- (4) *In particular, the line bundle $\mathcal{L}_{(\lambda, w^{-1}\lambda)}$ admits global sections on the closed substack $[(B^- \times B^-) \setminus \overline{C_w}]$ if and only if one has $\langle \lambda, ww_0\alpha^\vee \rangle \geq 0$ for all $\alpha \in E_w$.*

Proof. Assertion (1) is an elementary computation, which is left to the reader. Assertion (2) follows immediately using Lemma 4.1.1. Part (3) is Chevalley's formula, see for example [BL03] §1, page 654. Note that we are using a slightly different parametrization than in *loc. cit.*. The last part is a direct consequence of (3) and the normality of strata (Theorem 5.2.1). \square

5.3. Stratification of $G\text{-ZipFlag}^\mu$. The stack $G\text{-ZipFlag}^\mu$ carries a stratification parametrized by the Weyl group W of G . In the following, we will only use the realizations as quotient stacks of $G\text{-Zip}^\mu$ and $G\text{-ZipFlag}^\mu$. Furthermore, we will write equalities for the isomorphisms of Theorem 5.1.3, and identify the stacks. Recall that we defined $w_1 = w_0w_{0,I}$, where w_0 is the longest element in W and $w_{0,I}$ the longest element in W_I . If we denote by $w_{0,L}$ and $w_{0,M}$ the longest elements in the Weyl groups $W_L := W(L, T)$ and $W_M := W(M, T)$, then the following relations hold:

$$w_{0,L} = w_0w_{0,I}w_0 \quad \text{and} \quad w_{0,M} = \varphi(w_{0,L}) = w_0\varphi(w_{0,I})w_0.$$

Hence one has $w_1 = w_{0,L}w_0$ and $\varphi(w_1) = w_{0,M}w_0$. We consider the following morphism:

$$\psi : G \times L \rightarrow G, \quad (g, l) \mapsto l^{-1}g\varphi(l)\varphi(\dot{w}_1).$$

On the source of ψ , the group $E \times B_L^-$ acts on $G \times L$ as explained above. On the target of ψ , we consider the action of $B^- \times B^-$ on G given by $(u, v) \cdot g := ugv^{-1}$.

Proposition 5.3.1. *The map ψ induces a map of quotient stacks*

$$\psi : [(E \times B_L^-) \setminus (G \times L)] \rightarrow [(B^- \times B^-) \setminus G]$$

Proof. We define a map $\alpha : (E \times B_L^-) \times (G \times L) \rightarrow B^- \times B^-$ by

$$(((a, b), c), (g, l)) \mapsto (cl^{-1}\bar{a}^{-1}al, \varphi(\dot{w}_1)^{-1}\varphi(c)\varphi(l)^{-1}\varphi(\bar{a})^{-1}b\varphi(l)\varphi(\dot{w}_1)).$$

We show that the image of this map is indeed contained in $B^- \times B^-$. Note that the element $l^{-1}\bar{a}^{-1}al$ lies in R_uP , in particular in B^- . Similarly, the element $q := \varphi(l)^{-1}\varphi(\bar{a})^{-1}b\varphi(l)$ lies in R_uQ . The statement then follows from the inclusions:

$$\dot{w}_0\dot{w}_{0,M}B_M^-\dot{w}_{0,M}\dot{w}_0 = \dot{w}_0B_M\dot{w}_0 \subset B^-$$

$$\dot{w}_0\dot{w}_{0,M}R_u(Q)\dot{w}_{0,M}\dot{w}_0 = \dot{w}_0R_u(Q)\dot{w}_0 \subset B^-.$$

Now it can be checked immediately that the pair (ψ, α) satisfies the conditions of section §4.2, and hence induces a map of quotient stacks.

□

Proposition 5.3.2. *The morphism $\psi : G\text{-ZipFlag}^\mu \rightarrow [(B^- \times B^-) \backslash G]$ is smooth.*

Proof. It suffices to show that the map $\psi : G \times L \rightarrow G$ is smooth. This is clear since it is the composition of the group action morphism $(g, l) \mapsto l^{-1}g\varphi(l)$ with the translation by the element w_1 . □

We define the flag strata of the stack $G\text{-ZipFlag}^\mu$ as the fibers of the morphism ψ , endowed with the reduced structure. By Proposition 5.3.2 the flag strata are smooth. For an element $w \in W$, we define the set

$$H_w := \{(g, l) \in G \times L, \psi(g, l) \in C_w\}.$$

It is locally closed in $G \times L$, and we endow it with the reduced scheme structure. This scheme carries an action of the group $E \times B_L^-$ and the quotient stack

$$\mathcal{X}_w := [E \times B_L^- \backslash H_w]$$

is the flag stratum corresponding to $w \in W$. We define the closed flag strata attached to $w \in W$ as the inverse image by ψ of the substack $[(B^- \times B^-) \backslash \overline{C}_w]$.

Corollary 5.3.3.

- (i) *The closed flag strata are normal and irreducible.*
- (ii) *For an element $w \in W$, one has $\dim(H_w) = \ell(w) - \ell(w_0) + \dim(G \times L)$, where ℓ is the length function in W .*
- (iii) *The closed flag strata coincide with the closures of the flag strata.*

Proof. The first part of the first assertion follows from Theorem 5.2.1 and from the fact that normalization commutes with smooth base change. The morphism $\psi : G \times L \rightarrow G$ has irreducible fibers of the same dimension, C_w is irreducible and ψ is open, so $H_w = \psi^{-1}(C_w)$ is irreducible. Finally, the two last assertions follow from the smoothness of ψ . □

5.4. Images of flag strata. We now compute the images of flag strata by the morphism π . For an element $w \in W$, the set H_w is a union of $E \times B_L^-$ orbits. It follows that $\pi(H_w)$ is a union of E -orbits in G . Since H_w is irreducible, there is a unique E -orbit of maximal dimension in $\pi(H_w)$, and which we denote by $S_{\max}(w)$. In this way, we define a function:

$$(5.4.1) \quad S_{\max} : W \rightarrow \{E\text{-orbits in } G\}, \quad w \mapsto S_{\max}(w).$$

For two subsets $X, Y \subset G$, we write $X \sim Y$ if the set of E -orbits intersecting X coincides with the set of E -orbits intersecting Y . One has the following lemma:

Lemma 5.4.1. *For all $w \in W$, the set $\pi(H_w)$ is the set of E -orbits intersecting $B_L^- \dot{w} \varphi(\dot{w}_1)^{-1}$.*

Proof. By definition, $\pi(H_w)$ is the set of E -orbits intersecting $B^- \dot{w} B^- \varphi(\dot{w}_1)^{-1}$. One has $\varphi(\dot{w}_1) B^- \varphi(\dot{w}_1)^{-1} = B_M^- R_u(Q)$, so it follows

$$B^- w B^- \varphi(\dot{w}_1)^{-1} \sim B_L^- \dot{w} \varphi(\dot{w}_1)^{-1} B_M^- \sim B_L^- \dot{w} \varphi(\dot{w}_1)^{-1}.$$

□

We will now define minimal and cominimal elements in W . They have the property that $\pi(H_w)$ consists of a single E -orbit.

Definition 5.4.2. We define the set of minimal elements as $w_0^I W w_0 = w_1^I W$, where I is the type of P . We define the set of cominimal elements as $w_0 W^J w_0$, where J is the type of Q . If $w \in W$ is a minimal (resp. cominimal) element, we call the flag stratum \mathcal{X}_w a minimal (resp. cominimal) stratum.

Proposition 5.4.3. Let $w \in W$ be minimal or cominimal. Then $\pi(H_w)$ consists of a single E -orbit, namely the one containing $\dot{w}\varphi(\dot{w}_1)^{-1}$.

Proof. By Theorem 5.14 in [PWZ11], one has $\dot{w}_1 \dot{w}' B \sim \dot{w}_1 \dot{w}'$ for all $w' \in {}^I W$ and by the dual parametrization (Theorem 11.3 of *loc. cit.*) one has $\dot{w}_1 \dot{w}' B \sim \dot{w}_1 \dot{w}'$ for $w' \in W^J$. Let $w \in W$ be minimal or cominimal, and write $w = w_0 w' w_0$ with $w' \in {}^I W \cup W^J$. Then:

$$(5.4.2) \quad B^- \dot{w}\varphi(\dot{w}_1)^{-1} \sim B_L^- \dot{w}_0 \dot{w}' \dot{w}_{0,M} \sim \dot{w}_0 \dot{w}' \dot{w}_{0,M} B_M^-$$

The first equivalence is obtained by multiplying by elements in $R_u P$ on the left. In the second one, we move B_L^- to the right-hand side using elements of the form $(a, \varphi(a)) \in E$. Now this set is also equivalent to:

$$(5.4.3) \quad \dot{w}_0 \dot{w}' B_M \dot{w}_{0,M} \sim \dot{w}_{0,L} \dot{w}_0 \dot{w}' B_M \sim \dot{w}_1 \dot{w}' B_M \sim \dot{w}_1 \dot{w}'$$

and the assertion follows. \square

Note that the longest minimal (resp. cominimal) element is $w_0 w_{0,I} = w_1$ (resp. $w_{0,M} w_0 = \varphi(w_1)$). The identity element e is both the shortest minimal and cominimal element.

The previous proposition determines the value of the function S_{\max} at minimal and cominimal elements. For w minimal or cominimal, $S_{\max}(w)$ is the E -orbit of G containing $\dot{w}\varphi(\dot{w}_1)^{-1}$. As w varies in the set of minimal elements $w_0^I W w_0$, the value $S_{\max}(w)$ ranges over the E -orbits parameterized by $w_0^I W w_{0,M}$. This coincides with the set E -orbits parameterized by $w_1^I W$. By the parametrization of section §4.5, all E -orbits appear in this way. Using the dual parametrization, the same holds true for cominimal elements. Hence S_{\max} defines a bijection between the minimal (resp. cominimal) elements of W and the set of E -orbits in G . We have proved the following:

Proposition 5.4.4. For all E -orbits $S \subset G$, there is a unique minimal stratum H_w and a unique cominimal stratum H'_w such that $\pi(H_w) = \pi(H'_w) = S$.

We continue the study of minimal and cominimal flag strata.

Proposition 5.4.5. Assume $w \in W$ is minimal or cominimal. The variety H_w consists of a single orbit under the action of $E \times B_L^-$.

Proof. We will prove first that H_w contains finitely many $E \times B_L^-$ -orbits. We already know that H_w is irreducible, and that $\pi(H_w)$ consists of a single E -orbit. Hence for any element $x \in \pi(H_w)$, the fiber $\pi^{-1}(x) \cap H_w$ intersects all the $E \times B_L^-$ orbits in H_w . Furthermore the group B_L^- acts on this fiber. Therefore, it suffices to prove that this fiber consists of finitely many B_L^- -orbits. But clearly, the action of B_L^- is free, so it is enough to prove that the fiber of x in H_w has the same dimension as B_L^- . Since all fibers are isomorphic, the dimension of the fibers are $\dim(H_w) - \dim(\pi(H_w))$.

We have seen that the dimension of H_w varies continuously with the length of w (by this we mean that if the length of w drops by one, so does the dimension

of H_w). By the parametrizations given in section 4.5, the previous calculations show that the dimension of $S_{\max}(w)$ for w minimal (resp. cominimal) also satisfies this condition. Hence to show equality of the dimensions, it suffices to prove it for $w = e$. In this case, one has $S_{\max}(e) = E \cdot \dot{w}_1$. The minimal (and cominimal) stratum H_1 has dimension $\dim(G \times L) - \ell(w_0) = \dim(L) + \dim(B^-)$, and the E -orbit $S_{\max}(e)$ has dimension $\dim(P)$. It follows that the fibers have dimension $\dim(B_L^-)$, as claimed.

We have shown that there are finitely many $E \times B_L^-$ -orbits in H_w . Since H_w is irreducible, there exists an open dense orbit. Now assume that there is another orbit $Z \subset H_w$ of smaller dimension. Since it is an orbit, it maps surjectively onto $S_{\max}(w)$. But this contradicts the dimension formula for the fibers. Hence H_w consists of a single orbit. \square

Proposition 5.4.6. *Let $w \in W$ be minimal or cominimal and write $C := \pi(H_w)$. Then the preimage of C by the morphism $\pi : \overline{H_w} \rightarrow \overline{C}$ is exactly H_w .*

Proof. If some element in $\overline{H_w} \setminus H_w$ would be mapped to an element of C , then there would exist a flag stratum $\mathcal{X}_{w'}$ in the closure of \mathcal{X}_w , and a dominant map $\mathcal{X}_{w'} \rightarrow \overline{C}$ (since the image contains C , by E -equivariance). But this is clearly impossible because the dimension of $\mathcal{X}_{w'}$ is strictly smaller than the dimension of \mathcal{X}_w , which is equal to that of $[E \setminus C]$. \square

6. HASSE INVARIANTS

6.1. Line bundles on the stack of zip flags. As above we make the identifications $X^*(E) = X^*(P) = X^*(L)$. A character $(\lambda, \nu) \in X^*(L) \times X^*(T)$ thus induces a line bundle $\mathcal{M}_{(\lambda, \nu)}$ on the quotient stack $G\text{-ZipFlag}^\mu = [(E \times B_L^-) \setminus (G \times L)]$. For $\lambda \in X^*(L)$, the function

$$f_\lambda : G \times L \rightarrow \mathbf{G}_m, \quad (g, l) \mapsto \lambda(l)$$

satisfies the following relation: For $\epsilon = (a, b) \in E$, $c \in B_L^-$ and $(g, l) \in G \times L$, one has

$$f_\lambda((\epsilon, c) \cdot (g, l)) = \lambda(\overline{a}lc^{-1}) = \lambda(a)\lambda(c)^{-1}f_\lambda(g, l).$$

Since it has a non-vanishing global section, the line bundle $\mathcal{M}_{(\lambda, -\lambda)}$ is thus trivial. For all $(\lambda, \nu) \in X^*(L) \times X^*(T)$, we obtain an isomorphism of line bundles:

$$(6.1.1) \quad \mathcal{M}_{(\lambda, \nu)} \simeq \mathcal{M}_{(0, \lambda + \nu)}.$$

Hence we will only consider the line bundles $\mathcal{M}_\lambda := \mathcal{M}_{(0, \lambda)}$ for $\lambda \in X^*(T)$. For a character $(\lambda, \nu) \in X^*(T) \times X^*(T)$, we have a line bundle $\psi^* \mathcal{L}_{(\lambda, \nu)}$ on the stack of G -zip flags.

The action of the Weyl group W and the Galois group $\text{Gal}(k/\mathbf{F}_p)$ are related by the following formula. For all $w \in W$ and $\lambda \in X^*(T)$, one has

$$(6.1.2) \quad w(\lambda \circ \varphi) = (\varphi(w)\lambda) \circ \varphi.$$

The next proposition determines the pull-back of a line bundle by the map ψ . It follows from a simple computation, so we will omit the proof.

Proposition 6.1.1. *One has $\psi^* \mathcal{L}_{(\lambda, \nu)} = \mathcal{M}_\xi$ for $\xi = \lambda - (\varphi(w_1)\nu) \circ \varphi$.*

This computation leads us to introduce the map

$$(6.1.3) \quad h_w : X^*(T)_{\mathbf{Q}} \rightarrow X^*(T)_{\mathbf{Q}}, \quad \lambda \mapsto \lambda - (w\lambda) \circ \varphi$$

for an element $w \in W$.

For all $r \geq 1$, define $w^{(r)} := \varphi^{r-1}(w) \dots \varphi(w)w$. Using equation (6.1.2), one easily proves:

Lemma 6.1.2.

(i) Let $\lambda \in X^*(T)$ and $\alpha := h_w(\lambda)$. Then:

$$\alpha + (w\alpha) \circ \varphi + (w^{(2)}\alpha) \circ \varphi^2 + \dots + (w^{(r-1)}\alpha) \circ \varphi^{r-1} = \lambda - (w^{(r)}\lambda) \circ \varphi^r.$$

(ii) In particular, choose $r \geq 1$ such that $w^{(r)} = e$ and choose $m \geq 1$ such that λ is defined on \mathbf{F}_{p^m} . Then one has the relation:

$$\alpha + (w\alpha) \circ \varphi + (w^{(1)}\alpha) \circ \varphi^2 + \dots + (w^{(rm-1)}\alpha) \circ \varphi^{rm-1} = -(p^{rm} - 1)\lambda.$$

In this case, $w^{(rm-1)} = \varphi^{-1}(w)^{-1}$.

(iii) The map h_w is an isomorphism of \mathbf{Q} -vector spaces.

We now sketch our technique for constructing Hasse invariants, which will be carried out in detail below. For an element $w \in W$ and a character $\lambda \in X^*(T)$, consider the line bundle $\mathcal{L}_{\lambda, w^{-1}\lambda}$ on the Schubert stack. Its pullback to the stack $G\text{-ZipFlag}^\mu$ is the line bundle associated to the character $h_{\varphi(w_1)w^{-1}}(\lambda) = \lambda - (\varphi(w_1)w^{-1}\lambda) \circ \varphi$. By Theorem 5.2.2, we can determine when the line bundle $\mathcal{L}_{\lambda, w^{-1}\lambda}$ has global sections on the closed stratum corresponding to w . Pulling back along ψ , we can construct global sections on the closed flag strata.

6.2. Group-theoretical Hasse invariants. For a character $\chi \in X^*(L)$, denote by $\mathcal{D}(\chi)$ the associated line bundle on the stack $[E \backslash G]$. Recall the following result (Theorem 3.1 of [Kos14]):

Proposition 6.2.1. *Let C be an E -orbit in G . Then there exists an integer $N_C \geq 1$ such that for all characters $\chi \in X^*(L)$, the space $H^0([E \backslash C], \mathcal{D}(\chi)^{N_C})$ is one-dimensional.*

Any nonzero section $f \in H^0([E \backslash C], \mathcal{D}(\chi)^{N_C})$ identifies with a non-vanishing function $C \rightarrow \mathbf{G}_m$ satisfying the condition $f(\epsilon \cdot x) = \chi^{N_C}(\epsilon)f(x)$ for all $\epsilon \in E$ and $x \in C$.

Recall that a character $\chi \in X^*(L)$ is ample if it satisfies $\langle \chi, \alpha^\vee \rangle < 0$ for all $\alpha \in \Delta \setminus w_0 I w_0$ (see Definition 3.1.1).

Remark 6.2.2. The following statements are equivalent:

- (1) The character χ is ample as defined above.
- (2) The associated line bundle on G/P^+ is ample.
- (3) The associated line bundle on G/P is anti-ample.

The following result is Theorem 3.8 of [KW14]. As a consequence, one deduces the existence of μ -ordinary Hasse invariants on Shimura varieties of Hodge-type (Theorem 4.12 in *loc. cit.*).

Theorem 6.2.3 (Koskivirta-Wedhorn). *Let (G, μ) be a zip datum associated to a small¹⁴ cocharacter μ , and let $\chi \in X^*(L)$ be an ample character of L . Then there*

¹⁴Here “small” is meant in the sense of Definition 2.25 of [KW14].

exists an integer N such that the line bundle $\mathcal{D}(\chi)^{\otimes N}$ admits a section over the stack $G\text{-Zip}^\mu$, whose non-vanishing locus is exactly the open stratum $[E \setminus U] \subset G\text{-Zip}^\mu$, where $U \subset G$ denotes the unique open E -orbit.

We now prove Theorem 3.1.2, which is a considerable improvement of Theorem 6.2.3. We proceed in three steps. First, Lemma 6.2.4 establishes the existence of sections on closures of flag strata. Then Lemma 6.2.6 descends these sections to the normalizations of closures of E -orbits in G . Lemma 6.2.7 shows that the sections descend further to the E -orbit closures themselves. The latter argument employs a general principle about normalizations in positive characteristic.

Lemma 6.2.4. *Let $C \subset G$ be an E -orbit and let \mathcal{X} be the cominimal flag stratum lying above $[E \setminus C]$, given by Proposition 5.4.4. Denote by $\pi : \mathcal{X} \rightarrow [E \setminus C]$ the projection. Let $\chi \in X^*(L)$ be an ample and orbitally p -close character (see Definition 3.1.1). Then there exists an integer $N \geq 1$ and a section $f \in H^0([E \setminus C], \mathcal{D}(\chi)^{\otimes N})$, such that the section $\pi^* f$ of $\pi^* \mathcal{D}(\chi)^{\otimes N} = \mathcal{M}_\chi^{\otimes N}$ over \mathcal{X} extends to a section $\pi^* f \in H^0(\overline{\mathcal{X}}, \mathcal{M}_\chi^{\otimes N})$ whose non-vanishing locus is exactly the open substack \mathcal{X} . Furthermore, if \tilde{C} is the open E -orbit in G , the same result holds for any ample character χ .*

Proof. Replacing χ by a power, we may assume by Proposition 6.2.1 that there exists a nonzero section f over the stack $[E \setminus C]$ of $\mathcal{D}(\chi)$. The flag stratum \mathcal{X} corresponds to a cominimal element w such that $S_{\max}(w) = C$. Write $w = w_0 w' w_0$ with $w' \in W^J$. We claim that the section $\pi^* f$ of $\pi^* \mathcal{D}(\chi) = \mathcal{M}_\chi$ extends to a section over $\overline{\mathcal{X}}$.

Define $\lambda := h_{\varphi(w_1)w^{-1}}^{-1}(\chi) \in X^*(T)_{\mathbf{Q}}$. We may assume that $\lambda \in X^*(T)$ (since H_w is normal, we may without loss of generality replace f by a power of f). Now we choose a nonzero section $h_w \in H^0(C_w, \mathcal{L}_{\lambda, w^{-1}\lambda})$. By Proposition 6.1.1 one has $\psi^* h_w \in H^0(\mathcal{X}, \mathcal{M}_\chi)$. This space is one-dimensional, since H_w consists of a single $E \times B_L^-$ -orbit. Hence we may assume $\psi^* h_w = \pi^* f$. To show that $\pi^* f$ extends to $\overline{\mathcal{X}}$ and vanishes exactly on the complement of \mathcal{X} , it is thus equivalent to show that the Weil divisor of h_w is a linear combination with positive coefficients of the irreducible components of the complement of C_w in $\overline{C_w}$.

By Theorem 5.2.2, we have to show that $\langle \lambda, ww_0 \alpha^\vee \rangle > 0$ for all positive roots α satisfying $ww_0 s_\alpha w_0 < w$, or equivalently $w' s_\alpha < w'$. Now we use Lemma 6.1.2 for the element $\varphi(w_1)w^{-1}$. For r, m integers as in the lemma, one has:

$$(6.2.1) \quad \langle \lambda, ww_0 \alpha^\vee \rangle = -\frac{1}{p^m - 1} \sum_{i=0}^{rm-1} \langle ((\varphi(w_1)w^{-1})^{(i)} \chi) \circ \varphi^i, ww_0 \alpha^\vee \rangle.$$

The term corresponding to $i = rm - 1$ in (6.2.1) is equal to

$$(\varphi^{-1}(w)w_1^{-1}\chi) \circ \varphi^{rm-1}, ww_0 \alpha^\vee \rangle$$

by part (ii) of Lemma 6.1.2. Since χ is orbitally p -close and

$$(p-1) \sum_{i=0}^{rm-2} p^i = p^{rm-1} - 1 < p^{rm-1},$$

it suffices to show that

$$\langle (\varphi^{-1}(w)w_1^{-1}\chi) \circ \varphi^{rm-1}, ww_0 \alpha^\vee \rangle < 0$$

for all α as above. One has:

$$\begin{aligned}
\langle (\varphi^{-1}(w)w_1^{-1}\chi) \circ \varphi^{rm-1}, ww_0\alpha^\vee \rangle &= \langle w\varphi(w_1)^{-1}(\chi \circ \varphi^{rm-1}), ww_0\alpha^\vee \rangle \\
&= \langle w_{0,M}(\chi \circ \varphi^{rm-1}), \alpha^\vee \rangle \\
(6.2.2) \quad &= p^{rm-1} \langle \chi', w_{0,M}\alpha^\vee \rangle,
\end{aligned}$$

where χ' is defined by the equation $\chi' \circ \varphi = p\chi$.

Write $\alpha^\vee = \alpha_J^\vee + \alpha_{\Delta \setminus J}^\vee$, where α_J^\vee (resp. $\alpha_{\Delta \setminus J}^\vee$) is a nonnegative linear combination of elements of J^\vee (resp. $(\Delta \setminus J)^\vee$). Since $w's_\alpha < w'$, we know that $s_\alpha \notin W_J$. Therefore $s_{\alpha^\vee} \notin W_{J^\vee}$ and so $\alpha_{\Delta \setminus J}^\vee \neq 0$. Since $\langle \chi', w_{0,M}\alpha_J^\vee \rangle = 0$, we get that

$$(6.2.3) \quad \langle \chi', w_{0,M}\alpha^\vee \rangle = \langle \chi', w_{0,M}\alpha_{\Delta \setminus J}^\vee \rangle.$$

Since $w_{0,J}$ only alters the sign of simple roots which lie in J , the quantity (6.2.3) is negative, because χ is ample.

To finish the proof, assume C is the open E -orbit (so $w = \varphi(w_1)$) and χ is an ample character of L (no longer assumed to be orbitally p -close). Then formula 6.2.1 simplifies to:

$$(6.2.4) \quad \langle \lambda, ww_0\alpha^\vee \rangle = -\frac{1}{p^m - 1} \sum_{i=0}^{m-1} \langle \chi \circ \varphi^i, w_{0,M}\alpha^\vee \rangle.$$

Now the same argument as above shows that every summand of this sum is < 0 , hence the expression is > 0 , which concludes the proof. \square

Remark 6.2.5. The second part of the theorem is the "positivity conjecture" [KW14, Conjecture 3.6]. In *loc. cit.*, the positivity conjecture was only proved in the case of zip data arising from a small cocharacter (Theorem 3.8 of *loc. cit.*). In particular, Theorem 3.1.2 provides another proof of the existence of the μ -ordinary Hasse invariant on a Shimura variety of Hodge-type.

Lemma 6.2.6. *Let $C \subset G$ be an E -orbit, and denote by $\beta : \tilde{C} \rightarrow \overline{C}$ the normalization of its Zariski closure. Let $\chi \in X^*(L)$ be an ample and orbitally p -close character. Then there exists an integer $N \geq 1$ and a section $f \in H^0([E \setminus C], \mathcal{D}(\chi)^{\otimes N})$, such that the pullback β^*f extends to a section $\beta^*f \in H^0(\tilde{C}, \beta^*\mathcal{D}(\chi)^{\otimes N})$ whose non-vanishing locus is exactly the open substack $[E \setminus \beta^{-1}(C)]$.*

Proof. Keeping the notations of Lemma 6.2.4 and its proof, we obtain a section $f \in H^0([E \setminus C], \mathcal{D}(\chi)^{\otimes N})$, which we identify with a regular function $f : C \rightarrow k$ satisfying the relation $f(\epsilon \cdot x) = \chi^N(\epsilon)f(x)$ for all $\epsilon \in E$ and $x \in C$. Let w be the cominimal element such that $S_{\max}(w) = C$.

We have an E -equivariant map $\pi : B_L^- \setminus \overline{H_w} \rightarrow \overline{C}$ restricting to a map $\pi : H_w/B_L^- \rightarrow C$. The function $f \circ \pi$ extends (uniquely) to a regular function on $B_L^- \setminus \overline{H_w}$. Since $\overline{H_w}$ is normal, π factors through the normalization \tilde{C} . As π is proper, the induced map $\pi : B_L^- \setminus \overline{H_w} \rightarrow \tilde{C}$ is surjective. It follows immediately (by looking at Weil divisors) that the function f extends (uniquely) to \tilde{C} . The preimage $\pi^{-1}(C) \subset \overline{H_w}$ is precisely H_w by Proposition 5.4.6. Hence the non-vanishing locus of f in \tilde{C} is exactly $\beta^{-1}(C)$. \square

Lemma 6.2.7. *Let κ be a field of characteristic p , X an integral scheme of finite type over κ , and let $f : \tilde{X} \rightarrow X$ be the normalization of X . Let $U \subset X$ be an open subset such that $f : f^{-1}(U) \rightarrow U$ is an isomorphism. Denote by $Z = X - U$ the*

complement of U , endowed with the reduced structure. Let $s \in \mathcal{O}_{\tilde{X}}(\tilde{X})$ be a regular function which vanishes on $(Z \times_X Y)_{\text{red}}$. Then there exists $m \geq 1$, such that s^m descends to a section $s \in \mathcal{O}_X(X)$.

Proof. Clearly we may assume $X = \text{Spec } A$, and $\tilde{X} = \text{Spec } B$ where B is the integral closure of A in its fraction field. Let $I \subset A$ be the ideal of $Z \subset X$, so that

$$U = D(I) = \{\mathfrak{p} \in \text{Spec } A, I \not\subset \mathfrak{p}\}.$$

Replacing s by a power, we may assume that s vanishes on $Z \times_X Y$. Hence it lies in the ideal IB . Now we claim that there exists $n \geq 1$ such that s^{p^n} lies in A . Since κ has characteristic p , we may assume that $s = gx$ for some $g \in I$ and $x \in B$, because the p^n power map is additive. By our assumption that $f^{-1}(U) \rightarrow U$ is an isomorphism, it follows that the natural map $A_g \rightarrow B_g$ is an isomorphism (since $D(g) \subset U$). Hence we can find $m \geq 1$ such that $g^m x \in A$. Since $A[x]$ is generated as an A -module by $1, x, \dots, x^r$ for some $r \geq 1$, it follows that we can find m such that $g^m x^d \in A$ for all $d \geq 0$. Increasing m , we may assume that it is a power of p , say $m = p^n$. In particular $g^m x^m \in A$, and the proof is complete. \square

Proof of Theorem 3.1.2: Combine Lemmas 6.2.6 and 6.2.7. \square

Proof of Corollary 3.1.3: The ampleness of η_ω is straightforward, see [KW14, Remark 3.5]. The result then follows from Theorems 2.1.3 and 3.1.2, pulling back from $G\text{-Zip}^\mu$ to $\text{Sh}_\mathcal{K}^1$ along the morphism ζ . \square

Proof of Theorem 3.3.1, Part (1): By assumption, the special fiber $\text{Sh}_\mathcal{K}^1$ is proper. The Hodge line bundle ω is ample on $\text{Sh}_\mathcal{K}^1$. Hence part (1) of Theorem 3.3.1 is an immediate consequence of the general fact that the non-vanishing locus of a section of an ample line bundle on a projective scheme is affine. \square

6.3. Morphisms and Ekedahl-Oort strata. In this short subsection, we study the functoriality of Ekedahl-Oort strata. More precisely, if $f : S_1 \rightarrow S_2$ denotes the morphism induced on the special fibers of Shimura varieties by a finite morphism of Shimura data of Hodge-type, we want to understand how Ekedahl-Oort strata of S_1 map to strata of S_2 .

Let $f : (G_1, \mu_1) \rightarrow (G_2, \mu_2)$ be a morphism of zip data. Recall that f induces a morphism from the stacks of G_1 -zips to the stack of G_2 -zips. We say that f is finite if the morphism $f : G_1 \rightarrow G_2$ is finite. We denote by L_1 (resp. L_2) the Levi subgroups attached to μ_1 (resp. μ_2).

Theorem 6.3.1. *Let $f : (G_1, \mu_1) \rightarrow (G_2, \mu_2)$ be a finite morphism of zip data. Assume that there exists an ample, orbitally p -close character $\chi \in X^*(L_2)$, such that the restriction of χ to L_1 is again orbitally p -close (for G_1). Then the underlying map of topological spaces of the induced morphism of stacks*

$$(6.3.1) \quad G_1\text{-Zip}^{\mu_1} \rightarrow G_2\text{-Zip}^{\mu_2}$$

has discrete fibers.

Proof. Denote by P_1, P_2 the parabolics attached to μ_1, μ_2 . The map f induces a finite morphism $G/P_1 \rightarrow G/P_2$. It follows that the ample character χ restricts to an ample character $\chi \in X^*(L_1)$. Hence we may apply Theorem 3.1.2 to both stacks $G_1\text{-Zip}^{\mu_1}$ and $G_2\text{-Zip}^{\mu_2}$. Let C_1 denote a zip stratum for G_1 and let C_2 be the zip stratum for G_2 it maps to by f . There exists an integer $N \geq 1$ and a section

$H_i \in H^0(\overline{C_i}, \mathcal{D}(\chi)^{\otimes N})$ (for $i = 1, 2$), whose non-vanishing locus is C_i . Since these spaces of global sections are one-dimensional, the pull-back of H_2 to $\overline{C_1}$ agrees with H_1 up to a nonzero scalar. The result follows immediately. \square

By Theorem 2.1.3, we deduce the following corollary:

Corollary 6.3.2. *Let $\varphi : (\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(2g), \mathbf{X}_g)$ be an embedding of Shimura data, as in §2.1.4. Let (G, μ) be the attached zip datum, defined as in [KW14, §4.4]. Then the morphism of stacks $G\text{-Zip}^\mu \rightarrow GSp(2g)\text{-Zip}^{\mu_g}$ has discrete fibers. In other words, if two Ekedahl-Oort strata map to the same one under φ , then there is no closure relation between them.*

For example, if f is the natural embedding of a PEL-type Shimura variety in its Siegel Shimura variety, the naive stratification (defined by taking preimages of Siegel Ekedahl-Oort strata) is induced by the isomorphism class of the underlying BT_1 without its additional structure. Corollary 6.3.2 states that a naive Ekedahl-Oort strata is topologically a disjoint union of Ekedahl-Oort strata.

6.4. The cone of global sections. In this section, we look at the flag stratum \mathcal{X}_w corresponding to the element $w = w_0$, i.e. the open dense stratum of $G\text{-ZipFlag}^\mu$. We want to determine which line bundles admit global sections on this stack. By Proposition 6.1.1, the pullback of line bundles $\mathcal{L}_{\lambda, -w_0\lambda}$ from the stack of Schubert to the stack of zip flag is the line bundle \mathcal{M}_χ with $\chi = h_{w_0, M}(\lambda) = \lambda - (w_0, M\lambda) \circ \varphi$. If A is a subset of an abelian group, we denote by $\mathbf{Z}_{\geq 0}A$ the additive monoid generated by A . We define the following sets in $X^*(T)$:

- (1) \mathcal{C}_{sec} : the cone of characters $\lambda \in X^*(T)$ such that \mathcal{M}_λ admits nonzero global sections over $G\text{-ZipFlag}^\mu$.
- (2) \mathcal{C}_{++} : the image of the cone of dominant characters of T by the map $h_{w_0, M}$ above.
- (3) \mathcal{C} : the cone of characters $\lambda \in X^*(T)$ satisfying

$$\begin{aligned} \langle \lambda, \alpha^\vee \rangle &> 0 \text{ for } \alpha \in w_0 I w_0 \\ \langle \lambda, \alpha^\vee \rangle &< 0 \text{ for } \alpha \in \Delta \setminus w_0 I w_0 \end{aligned}$$

Equivalently, the character $w_0 w_0, I w_0 \lambda$ is regular anti-dominant.

- (4) A_p : the set of orbitally p -close characters in \mathcal{C} .

Proposition 6.4.1. *We have the inclusions:*

$$(6.4.1) \quad \mathbf{Z}_{\geq 0}A_p \subset \mathcal{C}_{++} \subset \mathcal{C}_{\text{sec}}$$

Proof. Let $\lambda \in A_p$ be a character. We have to show that $h_{w_0, M}^{-1}(\lambda)$ is G -dominant. Using the same notations for the integers r, m as in Lemma 6.1.2, we see that:

$$(6.4.2) \quad h_{w_0, M}^{-1}(\lambda) = -(p^{rm} - 1) \sum_{i=0}^{rm-1} (w_{0, M}^{(i)} \lambda) \circ \varphi^i$$

Since λ is orbitally p -close, it suffices to show that the term in the above sum corresponding to $i = rm - 1$ is regular anti-dominant. Using the second assertion of Lemma 6.1.2 (ii), this summand is $(\varphi^{-1}(w_{0, M})\lambda) \circ \varphi^{rm-1} = (w_0 w_0, I w_0 \lambda) \circ \varphi^{rm-1}$. Since the simple roots are defined over \mathbf{F}_p , the result follows from assumption that $w_0 w_0, I w_0 \lambda$ is regular anti-dominant. The second inclusion is clear. \square

7. EXTENDING COHOMOLOGICAL VANISHING TO STRATA

7.1. Preliminaries.

7.1.1. A “cohomology and base change” lemma.

Lemma 7.1.1. *Let (R, \mathfrak{m}_R) be a local ring, $f : X \rightarrow \operatorname{Spec} R$ a proper smooth morphism, and \mathcal{F} a coherent \mathcal{O}_X -module, flat over R . Denote by x the point $\mathfrak{m} \in \operatorname{Spec} R$. Assume that $H^i(X_x, \mathcal{F}_x) = 0$ for $i \geq 0$. Then $H^i(X, \mathcal{F}) = 0$.*

In particular, $H^i(X_n, \mathcal{F}_n) = 0$ for all $n \geq 1$, where $X_n = X \times_R R/\mathfrak{m}^n$ and \mathcal{F}_n is the pullback of \mathcal{F} via $X_n \rightarrow X$.

Proof. The base change map $\varphi^i(x) : R^i f_*(\mathcal{F})_x \otimes k(x) \rightarrow H^i(X_x, \mathcal{F}_x)$ is surjective, because its target is zero by assumption. By “cohomology and base change” (see [Har77, Theorem 12.11(a)] in the projective case and [Gro63, §7] in general¹⁵), the map $\varphi^i(x)$ is an isomorphism. Hence $R^i f_*(\mathcal{F})_x \otimes k(x) = 0$. Since $\operatorname{Spec} R$ is affine, $R^i f_*(\mathcal{F}) = H^i(X, \mathcal{F})^\sim$, and $H^i(X, \mathcal{F})$ is a finite type R -module since f is proper. So the first statement follows from Nakayama’s lemma.

The second part of the lemma follows from the first one applied to the base change $X_n \rightarrow \operatorname{Spec} R/\mathfrak{m}^n$. \square

7.1.2. Cohomological vanishing on $\operatorname{Sh}_{\mathcal{K}}^{\operatorname{tor}, n}$.

Lemma 7.1.2. *Let $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$. Assume at least one of the following three conditions hold:*

1. *The Shimura datum (\mathbf{G}, \mathbf{X}) is of PEL type.*
2. *$\eta \in \mathbf{Q}\eta_{\omega}$ and $p > \dim(\mathbf{G}, \mathbf{X})$.*
3. *For all $i > 0$, one has $R^i \pi_{1,*}^{\operatorname{tor}, \min} \mathcal{V}_{\eta}^{\operatorname{sub}} = 0$.*

Then for every $n \in \mathbf{Z}_{\geq 0}$, there exists a positive integer $m_0 = m_0(\eta, n)$, such that for all $m \geq m_0$ and all $i > 0$, one has

$$(7.1.1) \quad H^i(\operatorname{Sh}_{\mathcal{K}}^{\operatorname{tor}, n}, \mathcal{V}_{\eta}^{\operatorname{sub}} \otimes \omega^m) = 0.$$

Proof. By Lemma 7.1.1, it suffices to prove (7.1.1) when $n = 1$. If (\mathbf{G}, \mathbf{X}) is of PEL-type, condition 3. holds by a theorem of Lan [Lan, Th. 8.2.1.3]. Similarly, a result of Stroth [Str, Théorème 1] shows that condition 2. on η implies condition 3. on η .

So we may assume that η satisfies 3. This implies that the Leray spectral sequence degenerates for the map $\pi_1^{\operatorname{tor}, \min}$ and the sheaf $\mathcal{V}_{\eta}^{\operatorname{sub}} \otimes \omega^m$ (use the projection formula). Hence

$$(7.1.2) \quad H^i(\operatorname{Sh}_{\mathcal{K}}^{\operatorname{tor}, 1}, \mathcal{V}_{\eta}^{\operatorname{sub}} \otimes \omega^m) = H^i(\operatorname{Sh}_{\mathcal{K}}^{\min, 1}, \pi_{1,*}^{\operatorname{tor}, \min}(\mathcal{V}_{\eta}^{\operatorname{sub}} \otimes \omega^m)).$$

Since $(\pi_1^{\operatorname{tor}, \min})^* \omega_{\min} = \omega$, another application of the projection formula shows that the right-hand side of (7.1.2) simplifies to

$$(7.1.3) \quad H^i(\operatorname{Sh}_{\mathcal{K}}^{\min, 1}, (\pi_{1,*}^{\operatorname{tor}, \min} \mathcal{V}_{\eta}^{\operatorname{sub}}) \otimes \omega^m).$$

Since $\pi_1^{\operatorname{tor}, \min}$ is proper, $\pi_{1,*}^{\operatorname{tor}, \min} \mathcal{V}_{\eta}^{\operatorname{sub}}$ is coherent. Since ω_{\min} is ample on $\operatorname{Sh}_{\mathcal{K}}^{\min, 0}$, (7.1.3) is zero for all sufficiently large m by Serre’s cohomological characterization of ampleness. \square

¹⁵In the notes [Con], Conrad explains that, if one appears to [Gro61] for the coherence of higher direct images in the proper case, then the argument in [Har77] works in the proper case too.

A key point of this work is that in general the base change induced morphism

$$H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}, \mathcal{V}_{\eta}^{\mathrm{sub}}) \longrightarrow H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}, \mathcal{V}_{\eta}^{\mathrm{sub}})$$

is not surjective. However, the previous lemma implies that it becomes surjective if one twists $\mathcal{V}_{\eta}^{\mathrm{sub}}$ by a sufficiently large power of ω .

Corollary 7.1.3. *Keep the assumptions of Lemma 7.1.2 and let $m_0 = m_0(\eta, 0)$. Then for all $m \geq m_0$, the morphism*

$$(7.1.4) \quad H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}, \mathcal{V}_{\eta}^{\mathrm{sub}} \otimes \omega^m) \longrightarrow H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}, \mathcal{V}_{\eta}^{\mathrm{sub}} \otimes \omega^m)$$

is surjective.

Proof. Let ϖ be a uniformizer in $\mathcal{O}_{\mathfrak{p}}$. Multiplication by ϖ^n gives a short exact sequence

$$(7.1.5) \quad 0 \longrightarrow \mathcal{O}_{\mathfrak{p}} \longrightarrow \mathcal{O}_{\mathfrak{p}} \longrightarrow \mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n \longrightarrow 0.$$

The associated cohomology long exact sequence yields

$$(7.1.6) \quad H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}, \mathcal{V}_{\eta}^{\mathrm{sub}} \otimes \omega^m) \longrightarrow H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}, \mathcal{V}_{\eta}^{\mathrm{sub}} \otimes \omega^m) \longrightarrow H^1(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}, \mathcal{V}_{\eta}^{\mathrm{sub}} \otimes \omega^m).$$

By Lemma 7.1.2, when $m \geq m_0(\eta, 0)$, the right-hand term in (7.1.6) is zero. \square

7.2. Hecke-regular subschemes and the Cohen-Macaulay property.

Definition 7.2.1. *Let X be a scheme and \mathcal{L} a line bundle on X . We say that a global section $s \in H^0(X, \mathcal{L})$ is injective if the corresponding map of sheaves $\mathcal{O}_X \longrightarrow \mathcal{L}$ is injective.*

Remark 7.2.2. What we call an injective section is sometimes also referred to as a *regular section* (cf. [Sta15, Def. 11.17]). Since we find the terminology of *loc. cit.* confusing, we shall use the term *injective* instead.

Definition 7.2.3. *Let $X \subset \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}$ be a closed subscheme. Given a positive integer k , an regular sequence of length k in X consists of a sequence of triples $(Z_j, a_j, h_j)_{j=0}^k$ which satisfy:*

(HRS1) *We have $Z_0 = X$ and Z_{j+1} is a nonempty, proper closed subscheme of Z_j for all j , $0 \leq j \leq k-1$.*

(HRS2) *For all j , $0 \leq j \leq k$, one has an injective section $h_j \in H^0(Z_j, \omega^{a_j})$.*

(HRS3) *For all j , $0 \leq j \leq k-1$, the zero scheme of h_j is Z_{j+1} .*

If furthermore X is a Hecke-equivariant subscheme and the sections h_j are Hecke-equivariant (see remark below), we say that $(Z_j, a_j, h_j)_{j=0}^k$ is a Hecke-regular sequence.

Remark 7.2.4. Note that, if Z_j is Hecke-equivariant, it is meaningful to require that h_j be a Hecke-equivariant section, and if it is, then it induces a canonical Hecke-equivariant structure on its zero scheme Z_{j+1} . Thus the Hecke equivariance condition on the sections h_j is defined recursively.

Remark 7.2.5. By (HRS1), it is built in to the definition that the section h_j has a non-empty zero-scheme for all j , $0 \leq j \leq k-1$. On the other hand, it is possible (and indeed happens in our arguments) that h_k is nowhere vanishing.

Lemma 7.2.6. *Assume X is Cohen-Macaulay and let $(Z_j, a_j, h_j)_{j=0}^k$ be a regular sequence in X . Then Z_j is Cohen-Macaulay for all j , $0 \leq j \leq k$.*

Proof. The lemma follows by induction from the following three facts: (i) The zero scheme of an injective section is an effective Cartier divisor [Sta15, Lemma 30.11.20(4)]. (ii) If a locally Noetherian scheme satisfies Serre's property (S_k) , then any Cartier divisor in it satisfies (S_{k-1}) (Lemma 30.12.4 of *loc. cit.*). (iii) A scheme is Cohen-Macaulay if and only if it has Serre's property (S_k) for all $k \in \mathbf{Z}_{\geq 0}$ (Lemma 27.12.3 of *loc. cit.*). \square

Lemma 7.2.7. *Suppose $f : X \rightarrow S$ is a smooth morphism and S is Cohen-Macaulay. Then X is Cohen-Macaulay.*

Proof. Since the Cohen-Macaulay property is local [Eis95, Prop. 18.8] and since locally a smooth morphism factors via an étale morphism to an affine space, it is enough to consider the situation

$$(7.2.1) \quad \begin{array}{ccc} X & \xrightarrow{f_{\text{et}}} & \mathbf{A}_S^k \\ & \searrow f & \downarrow \\ & & S \end{array}$$

where f_{et} is étale.

By Proposition 18.9 of *loc. cit.*, \mathbf{A}_S^k is Cohen-Macaulay. Since the image of f_{et} is open, it too is Cohen-Macaulay. By [Sta15, Proposition 40.19.3], the source of an étale surjective morphism is Cohen-Macaulay if and only if its target is so. Hence X is Cohen-Macaulay. \square

Corollary 7.2.8. *For every $n \in \mathbf{Z}_{\geq 1}$, the scheme $\text{Sh}_{\mathcal{K}}^{\text{tor},n}$ is Cohen-Macaulay.*

Proof. The ring $\mathcal{O}_E/\mathfrak{p}^n$ is Cohen-Macaulay and the structure morphism $\text{Sh}_{\mathcal{K}}^{\text{tor},n} \rightarrow \text{Spec } \mathcal{O}_E/\mathfrak{p}^n$ is smooth, so the result follows from Lemma 7.2.7. \square

7.3. Cohomological vanishing for Hecke-regular subschemes of $\text{Sh}_{\mathcal{K}}^{\text{tor},n}$.

Lemma 7.3.1. *Suppose $\mathcal{Z} = (Z_j, a_j, h_j)_{j=0}^k$ is a Hecke-regular sequence with $Z_0 = \text{Sh}_{\mathcal{K}}^{\text{tor},n}$ and $\eta \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$. Assume that one of the three conditions of Lemma 7.1.2 holds. Then there exists an integer m_0 such that, for all $i > 0$, $m \geq m_0$, and $j \in \{0, \dots, k\}$, one has the vanishing*

$$(7.3.1) \quad H^i(Z_j, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^m) = 0.$$

Proof. The lemma is proved by induction on the length k of the Hecke-regular sequence. For all $r \in \mathbf{Z}_{\geq 0}$, multiplication by h_{k-1} gives a short exact sequence of sheaves on Z_{k-1} :

$$0 \rightarrow \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^r \rightarrow \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{a_{k-1}+r} \rightarrow \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{a_{k-1}+r}|_{Z_k} \rightarrow 0.$$

It induces a long exact sequence in cohomology:

$$(7.3.2) \quad H^i(Z_{k-1}, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{a_{k-1}+r}) \rightarrow H^i(Z_k, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{a_{k-1}+r}) \rightarrow H^{i+1}(Z_{k-1}, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^r)$$

By induction, the middle term is zero for sufficiently large r . \square

8. LIFTING INJECTIVE SECTIONS FOR p -NILPOTENT SCHEMES

8.1. The main lifting result.

Theorem 8.1.1. *Let X be a finite type $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n$ -scheme with special fiber X^1 . Let \mathcal{L} be a line bundle on X . Assume there is a decomposition*

$$(8.1.1) \quad (X^1)_{\text{red}} = X_1 \cup X_2 \cup \dots \cup X_r,$$

with each X_i closed reduced subscheme of X^1_{red} . There exists an integer m_0 such that for all $a \in \mathbf{Z}_{\geq 1}$ with $p^a \geq m_0$ and every r -tuple of sections $s_i \in H^0(X_i, \mathcal{L})$, $1 \leq i \leq r$, such that $s_i = s_j$ on $(X_i \cap X_j)_{\text{red}}$, there exists a unique section $s \in H^0(X, \mathcal{L}^{p^{2a}})$ whose restriction to X_i is $s_i^{p^{2a}}$ and which is Zariski-locally the p^a -power of a section s' of \mathcal{L}^{p^a} which restricts to s'_i on X_i . Moreover, if both of the schemes X and X^1 have no embedded components, then the section s is injective if and only if the sections s_1, \dots, s_r are all injective.

8.2. Schemes over a field of characteristic p . Let κ be a field of characteristic p . The following lemma may be abstracted from [ERX, §3.3.1].

Lemma 8.2.1. *Let X be a finite type κ -scheme. Let \mathcal{I} be the ideal sheaf of X_{red} in X and let m_0 denote its nilpotence degree. Suppose \mathcal{L} is a line bundle on X and $s \in H^0(X_{\text{red}}, \mathcal{L})$. Then for all $a \in \mathbf{Z}_{\geq 1}$ with $p^a \geq m_0$, there exists a unique section $t \in H^0(X, \mathcal{L}^{p^a})$ which is Zariski-locally on X the p^a th power of a lift of s (in particular the restriction of t to X_{red} is s^{p^a}).*

Proof. Assume first that $X = \text{Spec } A$ is affine. Let I denote the ideal of A corresponding to the ideal sheaf \mathcal{I} . If $s_1, s_2 \in A$ are two lifts of $s \in A/I$, then $s_1 - s_2 \in I$, so $(s_1 - s_2)^m = 0$ for all $m \geq m_0$. But $(s_1 - s_2)^{p^a} = s_1^{p^a} - s_2^{p^a}$, since A is a k -algebra and k has characteristic p . Hence $s_1^{p^a} = s_2^{p^a}$ when $p^a \geq m_0$.

Returning to the general case, the above shows that, when $p^a \geq m_0$, the p^a -th powers of local lifts of s glue to give a global lift of s^{p^a} to X , which is the unique lift that is locally the p^a -th power of a lift of s . \square

Lemma 8.2.2. *Let X be a reduced, finite type κ -scheme and let \mathcal{L} be a line bundle on X . Suppose $X = X_1 \cup X_2$, with X_1 and X_2 closed subschemes of X . Let \mathcal{I}_{12} be the ideal sheaf of $(X_1 \cap X_2)_{\text{red}}$ in $X_1 \cap X_2$ and denote its nilpotence degree by m_0 . For every pair of sections $s_1 \in H^0(X_1, \mathcal{L})$ and $s_2 \in H^0(X_2, \mathcal{L})$ which agree on $(X_1 \cap X_2)_{\text{red}}$, and every $a \in \mathbf{Z}_{\geq 1}$ with $p^a \geq m_0$, there exists a unique section $s \in H^0(X, \mathcal{L}^{p^a})$ which restricts to $s_1^{p^a}$ on X_1 and $s_2^{p^a}$ on X_2 .*

Proof. We may assume that $X = \text{Spec } A$, $X_1 = \text{Spec } A/I_1$, $X_2 = \text{Spec } A/I_2$, $\mathcal{L} = \mathcal{O}_X$ since the general case is then handled by gluing. Since A is reduced, we have the short exact sequence

$$(8.2.1) \quad 0 \longrightarrow A \longrightarrow A/I_1 \oplus A/I_2 \longrightarrow A/(I_1 + I_2) \longrightarrow 0$$

where the second map is $(g_1, g_2) \mapsto g_1 - g_2$. Let I_{12} be the ideal of nilpotent elements of $A/(I_1 + I_2)$. By assumption, $s_1 - s_2 \in I_{12}$. As in the proof of Lemma 8.2.1, one has $s_1^{p^a} = s_2^{p^a}$ in $A/(I_1 + I_2)$ for all a such that $p^a \geq m_0$. Therefore there exists a unique element $s \in A$ mapping to $(s_1^{p^a}, s_2^{p^a})$. \square

Corollary 8.2.3. *Let X be a finite type κ -scheme and let \mathcal{L} be a line bundle on X . Suppose $X = X_1 \cup \dots \cup X_r$, where each X_i is a closed subscheme of X . Let $s_i \in H^0(X_i, \mathcal{L})$ for $i = 1, \dots, r$ be sections such that s_i and s_j agree on $(X_i \cap X_j)_{\text{red}}$. Then there exists an integer m_0 such that for all $a \in \mathbf{Z}_{\geq 1}$ with $p^a \geq m_0$, there exists a section $t \in H^0(X, \mathcal{L}^{p^a})$ such that the restriction of t to X_i is $s_i^{p^a}$.*

Proof. Clearly we may assume $r = 2$. Consider $X_{\text{red}} = X_{1,\text{red}} \cup X_{2,\text{red}}$ and let $s'_i \in H^0(X_{i,\text{red}}, \mathcal{L})$ be the restriction of s_i . Choose m_0 larger than the boundaries given by lemma 8.2.3 for $X_{1,\text{red}} \subset X_1$, $X_{2,\text{red}} \subset X_2$ and $X_{\text{red}} \subset X$, and larger than the boundary given by lemma 8.2.2 for the decomposition $X_{\text{red}} = X_{1,\text{red}} \cup X_{2,\text{red}}$. Now, let $a \in \mathbf{Z}_{\geq 1}$ be an integer with $p^a \geq m_0$. There exists a (unique) $s' \in H^0(X_{\text{red}}, \mathcal{L}^{p^a})$ restricting to $s_i'^{p^a}$ on $X_{i,\text{red}}$, and there exists a unique section $t \in H^0(X, \mathcal{L}^{p^{2a}})$ which is locally the p^a th power of a lift of s' . Now $t|_{X_i}$ and $s_i'^{p^{2a}}$ are both locally the p^a power of a lift of $s_i'^{p^a}$, so they must agree. \square

8.3. Schemes over $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n$.

Lemma 8.3.1. *Let X be a finite type $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n$ -scheme with special fiber X^1 . Suppose \mathcal{L} is a line bundle on X and $s \in H^0(X^1, \mathcal{L})$. Then for all $a \in \mathbf{Z}_{\geq 1}$ with $s^{p^{(n-1)a}}$ there exists a unique section $t \in H^0(X, \mathcal{L}^{p^{n-1}a})$ which is locally the $p^{n-1}a$ -th power of a lift of s .*

Proof. We may assume $X = \text{Spec } A$ for some $\mathcal{O}_{\mathfrak{p}}/\mathfrak{p}^n$ -algebra A , and $\mathcal{L} = \mathcal{O}_X$. Then $X^1 = \text{Spec } A/pA$. For $s, x \in A$, one has for all $r \geq 1$, $(s + px)^{p^{r-1}} \equiv s^{p^{r-1}} \pmod{p^r}$, hence $(s + px)^{p^{n-1}} = s^{p^{n-1}}$. Hence the p^{n-1} -th power of a lift of s is independent of the choice of this lift. The result follows. \square

8.4. Embedded components and injectivity.

Lemma 8.4.1. *Let X be a Noetherian scheme without embedded components. Let \mathcal{L} be a line bundle on X . Suppose $X' \rightarrow X$ is a closed embedding defined by a nilpotent ideal sheaf \mathcal{I} , and let $s \in H^0(X, \mathcal{L})$ with restriction $s' \in H^0(X', \mathcal{L})$. Then:*

- (1) *If s' is injective, then s is injective.*
- (2) *Assume that X' has no embedded components. Then s is injective if and only if s' is injective.*
- (3) *In particular, when $X' = X_{\text{red}}$, the section s is injective if and only if s' is injective.*

Proof. We may assume $X = \text{Spec } A$, $X' = \text{Spec } A/I$ and $\mathcal{L} = \mathcal{O}_X$. Let $s \in A$ such that $\overline{s} \in A/I$ is a regular element, and let $x \in R$ such that $sx = 0$. Since X has no embedded components, there is a primary decomposition

$$(0) = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r$$

such that the ideals $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ are pairwise distinct and are exactly the minimal primes of A . In particular, one has:

$$\sqrt{(0)} = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r.$$

If x is nonzero, then there exists $j \in \{1, \dots, r\}$ such that $x \notin \mathfrak{q}_j$, and we may assume $j = 1$. Hence some power of s lies in \mathfrak{p}_1 , and replacing s by this power, we may assume $s \in \mathfrak{p}_1$. Now choose $s_i \in \mathfrak{p}_i$ for $i = 2, \dots, r$ such that $s_2 \dots s_r \notin \sqrt{(0)}$. This is

clearly possible, since otherwise one would have $\sqrt{(0)} = \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_r$ which contradicts the fact that the \mathfrak{p}_i are pairwise distinct. Now $ss_1 \dots s_r \in \sqrt{(0)}$, so there exists $n \geq 1$ such that $s^n(s_2 \dots s_r)^n = 0$. By assumption, this implies $(s_2 \dots s_r)^n \in I \subset \sqrt{(0)}$, so $s_2 \dots s_r \in \sqrt{(0)}$, which is a contradiction. It follows that $x = 0$ and so s is a regular element of A . For the second part of the lemma, assume that $s \in A$ is a regular element. If $sx \in \sqrt{(0)}$, then it follows $s^n x^n = 0$ for some $n \geq 1$, thus $x^n = 0$, which completes the proof. \square

Proof of Th. 8.1.1: Combining Cor. 8.2.3 and Lemma 8.3.1 gives the first part of the theorem. To prove the statement on injectivity, notice that if X is a $\mathcal{O}_p/\mathfrak{p}^n$ -scheme, then $(X^1)_{\text{red}} = X_{\text{red}}$. Then we can apply Lemma 8.4.1 (3). \square

9. EXTENDING THE EKEDAH-LOORT STRATIFICATION TO TOROIDAL COMPACTIFICATIONS

9.1. Extending the universal G -Zip. Let φ be an embedding of Shimura data as in (2.1.2). From (2.1.4), the sheaf $\varphi^* H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$ is a rank $2g$, locally free extension of $\varphi^* H_{\text{dR}}^1(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$ to $\text{Sh}_{\mathcal{K}}^{\text{tor},1}$. Let $\tilde{\mathcal{A}}^1$ be the universal semi-abelian scheme over $\text{Sh}_{g,\mathcal{K}}^{\text{tor},1}$ and let $\tilde{\Omega}_g$ be the pull-back along the identity section of the relative differentials on $\tilde{\mathcal{A}}^1$. Let $\overline{\mathcal{A}}$ be the compactification of $\tilde{\mathcal{A}}$ constructed in [CF90, §6.1] and [Lan12, §2.B].

By *loc. cit.*, the sheaf $H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$ is identified with the log de Rham cohomology $H_{\log-\text{dR}}^1(\overline{\mathcal{A}}^1/\text{Sh}_{g,\mathcal{K}}^1)$. Moreover, for all $j \in \mathbf{Z}_{\geq 1}$, we have an identification

$$(9.1.1) \quad H_{\log-\text{dR}}^j(\overline{\mathcal{A}}^1/\text{Sh}_{g,\mathcal{K}}^1) \cong \wedge^j H_{\log-\text{dR}}^1(\overline{\mathcal{A}}^1/\text{Sh}_{g,\mathcal{K}}^1).$$

Hence we can appeal to Theorem 8.0 of [Kat70] to see that the log de Rham cohomology of $\overline{\mathcal{A}}^1/\text{Sh}_{g,\mathcal{K}}^1$ commutes with arbitrary base change. In particular, $(H_{\log-\text{dR}}^1(\overline{\mathcal{A}}^1/\text{Sh}_{g,\mathcal{K}}^1))^{(p)}$ is canonically isomorphic to $H_{\log-\text{dR}}^1((\overline{\mathcal{A}}^1)^{(p)}/\text{Sh}_{g,\mathcal{K}}^1)$. Let

$$(9.1.2) \quad F : (H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1))^{(p)} \rightarrow H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$$

denote the map induced by the relative Frobenius of $\overline{\mathcal{A}}$ over $\text{Sh}_{g,\mathcal{K}}^{\text{tor},1}$.

The natural pairing on $H_{\text{dR}}^1(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$ extends (uniquely) to a perfect pairing on $H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$, for which $\tilde{\Omega}_g$ is a maximal, totally isotropic, locally direct factor. The pairing on $H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$ induces one on $(H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1))^{(p)}$ which is also perfect.

We define Verschiebung

$$(9.1.3) \quad V : H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1) \rightarrow H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)^{(p)}$$

as the transpose of F relative to the perfect pairings mentioned above. This provides an extension (which is necessarily unique) of the classical $V : H_{\text{dR}}^1(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1) \rightarrow H_{\text{dR}}^1(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)^{(p)}$ induced by the isogeny $V : (\mathcal{A}^1)^{(p)} \rightarrow \mathcal{A}^1$.

Since the relations $FV = 0$ and $VF = 0$ hold over $\text{Sh}_{g,\mathcal{K}}^1$, which is open dense in $\text{Sh}_{g,\mathcal{K}}^{\text{tor},1}$, they continue to hold over $\text{Sh}_{g,\mathcal{K}}^{\text{tor},1}$. To simplify notations, write $\mathcal{M} = H_{\text{dR}}^{1,\text{can}}(\mathcal{A}^1/\text{Sh}_{g,\mathcal{K}}^1)$. For all $x \in \text{Sh}_{\mathcal{K}}^{\text{tor},0}$, we get induced morphisms of $k(x)$ -vector spaces $F_x : \mathcal{M}^{(p)} \otimes k(x) \rightarrow \mathcal{M} \otimes k(x)$ and $V_x : \mathcal{M} \otimes k(x) \rightarrow \mathcal{M}^{(p)} \otimes k(x)$ which are again transpose to each other with respect to the pairing. It is clear that

$(\mathrm{Im} V) \otimes k(x) = \mathrm{Im} V_x$ and $(\mathrm{Im} F) \otimes k(x) = \mathrm{Im} F_x$, since pull-back is right-exact. It follows that $(\mathrm{Im} V) \otimes k(x)$ and $(\mathrm{Im} F) \otimes k(x)$ have constant rank g . Since $\mathrm{Sh}_{g,\mathcal{K}}^{\mathrm{tor},1}$ is reduced, $\mathrm{Im} F$ and $\mathrm{Im} V$ are locally free of rank g . The exact sequence $0 \rightarrow \ker F \rightarrow M^{(p)} \rightarrow \mathrm{Im} F \rightarrow 0$ is thus locally split, so $\ker F$ and $\ker V$ are locally free of rank g as well. It also implies that $(\ker F) \otimes k(x) = \ker F_x$, so we conclude $\ker F = \mathrm{Im} V$ by Nakayama's lemma, and similarly $\ker V = \mathrm{Im} F$.

On the other hand, it follows from the definition of $\tilde{\Omega}_g$ that $\tilde{\Omega}_g^{(p)}$ is contained in $\ker F$. Since both sheaves are locally direct factors, we deduce $\tilde{\Omega}_g^{(p)} = \ker F$.

We now have two morphisms $\tilde{\Omega}_g \rightarrow \tilde{\Omega}_g^{(p)}$. The first is given by restricting (9.1.3) to $\tilde{\Omega}_g$, noting that the image of (9.1.3) lands in $\tilde{\Omega}_g^{(p)}$. The second is given as the map induced on differentials by the Verschiebung of the semi-abelian scheme $\tilde{\mathcal{A}}^1$. It is clear that these two morphisms agree on the complement of the boundary, hence they agree over $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$.

Recall that we defined a P^0 -torsor \mathfrak{P}^0 in §2.1.5, following [Mad]. Let Q^1 be the Frobenius pull-back of the parabolic opposite to P^1 . Also write Q_g^1 in the Siegel case. We claim that, by replacing de Rham homology with cohomology and replacing the Hodge filtration with the filtration $0 \subset \ker V \subset H_{\mathrm{dR}}^{1,\mathrm{can}}(\mathcal{A}^1/\mathrm{Sh}_{g,\mathcal{K}}^1)$ in the definition of \mathfrak{P}^1 , we get a Q^1 -torsor Ω^1 over $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$, which extends the one defined by Zhang in Theorem 2.4.1, 3) of [Zha]. Given the above discussion about F and V , it is clear that the claim is true when $(\mathbf{G}, \mathbf{X}) = (GSp(2g), \mathbf{X}_g)$.

The delicate point is to show that Ω^1 is étale locally trivial. We thank Torsten Wedhorn for his help with this. The key is to show that, for $x \in \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}$, the type of the parabolic stabilizing the ‘conjugate’ filtration $0 \subset (\ker V) \otimes k(x) \subset H_{\mathrm{dR}}^{1,\mathrm{can}}(\mathcal{A}^1/\mathrm{Sh}_{g,\mathcal{K}}^1) \otimes k(x)$ is a constant function of x on $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}$. By *loc. cit.*, the type is constant on $\mathrm{Sh}_{\mathcal{K}}^1$. But the type is a locally constant function, so we deduce that it is constant on all of $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}$.

Proof of Lemma 3.2.1: Put $\mathcal{I}^{\mathrm{tor}} = (\mathfrak{G}^1, \mathfrak{P}^1, \Omega^1, F)$, where F is induced from (9.1.2). The above discussion shows that $\mathcal{I}^{\mathrm{tor}}$ is a G -Zip over $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$, which extends the G -Zip \mathcal{I} over $\mathrm{Sh}_{\mathcal{K}}^1$ defined by Zhang in *loc. cit.* \square

9.2. Length stratification. Let \mathcal{I} be a G -zip of type μ over a scheme S of finite type over κ . It induces a morphism of stacks $f : S \rightarrow G\text{-Zip}^\mu$. This defines a locally-closed stratification of S :

$$(9.2.1) \quad S = \bigsqcup_{w \in {}^I W} S_w$$

where we endow the locally closed subsets $S_w = f^{-1}(w)$ with the reduced scheme structure. We define:

$$(9.2.2) \quad \hat{S}_w = \bigsqcup_{w' \in \overline{w}} S_{w'}$$

It is clear that \hat{S}_w is closed, and S_w is open in \hat{S}_w . Note that it is false in general that \hat{S}_w is the Zariski closure of S_w in S . The smallest stratum in S is S_e (i.e. for $w = e$). Let d denote the length of the maximal element $w_{0,I} w_0 \in {}^I W$. For $j \in \{0, \dots, d\}$, define:

$$(9.2.3) \quad S_j = \bigsqcup_{\ell(w)=j} S_w$$

$$(9.2.4) \quad \hat{S}_j = \bigsqcup_{\ell(w) \leq j} S_w$$

Again, we endow these locally closed subsets with the reduced scheme structure.

Definition 9.2.1. *We call S_j the j -th length stratum of S .*

If $w = s_{\alpha_1} \cdots s_{\alpha_r}$ is a reduced expression of an element $w \in {}^I W$, then $w' := s_{\alpha_1} \cdots s_{\alpha_{r-1}}$ is again an element in ${}^I W$ of length $r - 1$. Hence any element in ${}^I W$ of length $r \geq 1$ has an element of ${}^I W$ of length $r - 1$ in its closure. But since $w \mapsto w_0 I w w_0$ is an order-reversing involution, we deduce similarly that any element of length $\leq j$ in ${}^I W$ lies in the closure of an element of length j . It follows that \hat{S}_j decomposes as a union:

$$(9.2.5) \quad \hat{S}_j = \bigcup_{\ell(w)=j} \hat{S}_w$$

Using Theorem 3.1.2, we can find $m \geq 1$ and sections $h_w \in H^0(\hat{S}_w, \mathcal{D}(\chi)^{\otimes m})$ such that the non-vanishing locus of h_w in \hat{S}_w is S_w . By Theorem 8.1.1, we can glue the sections h_w for $\ell(w) = j$ to obtain sections $h_j \in H^0(\hat{S}_j, \mathcal{D}(\chi)^{\otimes m'})$, for some integer $m' \geq m$. The non-vanishing locus of h_j is then exactly S_j .

Proposition 9.2.2. *Let S be a finite-type scheme over κ . Assume the following:*

- (1) *The scheme S is equi-dimensional of dimension d .*
- (2) *The stratum S_w is non-empty, for all $w \in {}^I W$.*
- (3) *The stratum S_0 is zero-dimensional.*

Then S_j and \hat{S}_j are equi-dimensional of dimension j , and S_j is open dense in \hat{S}_j . For $w \in {}^I W$, S_w is equi-dimensional of dimension j .

Remark 9.2.3. Under the assumptions of Proposition 9.2.2, we do not claim that S_w is dense in \hat{S}_w , nor that \hat{S}_w is equi-dimensional.

Proof. Since \hat{S}_{j-1} is cut out by the section h_j in \hat{S}_j , we deduce that

$$(9.2.6) \quad \dim(\hat{S}_j) - 1 \leq \dim(\hat{S}_{j-1}) \leq \dim(\hat{S}_j)$$

The assumptions (1) and (3) then imply that $\dim(\hat{S}_j) = j$ for all $j = 0, \dots, d$. We now prove by decreasing induction on j that \hat{S}_j is equi-dimensional of dimension j . This is true for $j = d$ by assumption (1). Assume it is true for $j \geq 1$. Since $\dim \hat{S}_{j-1} = j - 1$, no irreducible component of \hat{S}_j is contained in \hat{S}_{j-1} . Hence the open subset $S_j \subset \hat{S}_j$ intersects all irreducible components of \hat{S}_j . This means the closed subscheme \hat{S}_{j-1} is cut out by an injective section, so it is pure of dimension $j - 1$. Considering dimensions again, it follows that S_j is dense in \hat{S}_j .

If a stratum S_w for $\ell(w) \leq j$ has dimension j , then $\ell(w) = j$. Conversely, let $w \in W$ be an element of length j . There exists an element w' of length j satisfying $\dim(\hat{S}_{w'}) = j$ such that S_w intersects $\hat{S}_{w'}$, because we have proved that \hat{S}_j is pure of dimension j . But the continuity of f implies $w' = w$, so $\dim(\hat{S}_w) = j$. Hence $\dim(S_w) = j$ as well. Let Z be an irreducible component of S_w , and assume $\dim(Z) < j$. Then Z is contained in $\hat{S}_{w'}$ for an element $w' \neq w$ of length j , because \hat{S}_j is pure of dimension j . Again, this contradicts the continuity of f . Hence S_w is pure of dimension j . \square

9.3. The length stratification of Hodge-type Shimura varieties. We now restrict the general scheme S of §9.2 to be of the form $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$. Assumption (1) of Proposition 9.2.2 follows from [Mad, Theorem 2]. Assumption (2) was proved in the PEL case in [VW13, Theorem 2], and has recently been generalized to general Hodge-type Shimura varieties, see [Yu],[Kis], [Lee]. We thank Adrian Vasiu for communicating to us that his paper [Vas11] proved Assumption (2) in the general Hodge case using a different language from the more recent articles cited above. As for Assumption (3), we have the following lemma:

Lemma 9.3.1. *Suppose $S = \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ is the special fiber of a Shimura variety of PEL type. Then the smallest stratum S_e^{tor} intersects the boundary trivially: $S_e = S_e^{\mathrm{tor}}$.*

Proof. The lemma is trivial when $Sh(\mathbf{G}, \mathbf{X})$ is of compact type. Thus we assume $Sh(\mathbf{G}, \mathbf{X})$ is not of compact type. By [BB66, Lemma 3.2(b)], this implies that $\mathbf{G}^{\mathrm{ad}}(\mathbf{R})$ has no compact factors. Since $S_e \subset S_e^{\mathrm{tor}}$, the latter is nonempty as well. By choosing compatible toroidal compactifications, we get a map $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1} \rightarrow \mathrm{Sh}_{g,\mathcal{K}}^{\mathrm{tor},1}$, extending the canonical map $\mathrm{Sh}_{\mathcal{K}}^1 \rightarrow \mathrm{Sh}_{g,\mathcal{K}}^1$ induced by the forgetful functor. The image of S_e^{tor} is contained in a unique Siegel stratum $S_{g,w}^{\mathrm{tor}}$ (for a unique $w \in W_g$). So it suffices to show that $S_{g,w}^{\mathrm{tor}}$ does not meet the boundary of $\mathrm{Sh}_{g,\mathcal{K}}^{\mathrm{tor},1}$. The result now follows from the following facts:

- (1) The p -rank of the semi-abelian variety corresponding to any boundary point of $\mathrm{Sh}_{g,\mathcal{K}}^{\mathrm{tor},1}$ is > 0 .
- (2) The p -rank is constant along the strata $S_{g,w}^{\mathrm{tor}}$ for $w \in W_g$.
- (3) The stratum $S_{g,w}^{\mathrm{tor}}$ containing S_e^{tor} has p -rank 0.

The first fact can be seen as follows: The p -rank of a semi-abelian variety is equal to the semisimple rank s of V acting on its differentials. At a point $x \in \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ where the torus part has dimension j , we have $s \geq j$, because V induces an isomorphism on the differentials of \mathcal{A}_x^1 modulo the differentials of its abelian part. Hence $s > 0$ for all x in the boundary.

Now consider (2). In view of the transpose relation between F and V , we know that s is also the semisimple rank of $F : (\ker V)^{(p)} \rightarrow \ker V$. Since the semisimple rank is equal to the stable rank (i.e., the rank of all sufficiently large powers of F), we see that s is an invariant of the G -Zip $\mathcal{I}_x^{\mathrm{tor}}$.

To prove (3), (2) shows that it is enough to show that the p -rank of the underlying abelian variety of a point of S_e is zero, under the assumption that $\mathbf{G}^{\mathrm{ad}}(\mathbf{R})$ has no compact factors. This can be checked case-by-case using Moonen's "standard objects" [Moo01, §§4.9,5.8]. (Given $w \in {}^I W$, the associated standard object is an explicit representative of the isomorphism class of the G -Zips of type w .) \square

10. CONSTRUCTION OF HECKE FACTORIZATIONS

In §10 we prove Theorem 3.4.1 and Corollary 3.4.2. The proof of Theorem 3.4.1 naturally breaks down into three steps, which we call (i) Going down, (ii) Increasing the weight and (iii) Going up. Step (i) produces a factorization as in (3.4.1), but where the Hecke algebra in the top right is one associated to a zero-dimensional subscheme of $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}$ whose associated reduced scheme is the zero-dimensional Ekedahl-Oort stratum. Step (ii) then shows that the projection to this Hecke algebra factors through a Hecke algebra for the same zero-dimensional scheme, but where the

weight of the automorphic vector bundle has increased sufficiently so that the vanishing lemma 7.3.1 is applicable. Step (iii) shows that the projection to the Hecke algebra in Step (ii) factors through the Hecke algebra in the top right of (3.4.1). Composing these factorizations yields Theorem 3.4.1.

Following the proof of Theorem 3.4.1, we introduce the flag space in §10.3.1. Using generalized Hasse invariants on the flag space, we deduce Corollary 3.4.2 in §10.3.2.

10.1. Preliminaries.

Lemma 10.1.1. *Suppose R is a \mathbf{Z}_p -algebra and $\epsilon : M' \rightarrow M$ is a morphism of $R\mathcal{H}$ -modules. Writing \mathcal{H}_M (resp. $\mathcal{H}_{M'}$) for the image of \mathcal{H} in $\text{End}_R(M)$ (resp. $\text{End}_R(M')$), we have the following:*

1. *If ϵ is injective, then there exists a factorization:*

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_M \\ \downarrow & \swarrow & \\ \mathcal{H}_{M'} & & \end{array}$$

2. *If ϵ is surjective, then there exists a factorization:*

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{M'} \\ \downarrow & \swarrow & \\ \mathcal{H}_M & & \end{array}$$

Proof. Both factorizations are equivalent to the corresponding inclusions of kernels, which are trivial to verify. \square

The key objects in the proof of Theorem 3.4.1 are particular Hecke regular sequences which are adapted to the Ekedahl-Oort stratification. We make this property precise in the following:

Definition 10.1.2. *Let $m \in \mathbf{Z}_{\geq 0}$ and assume $m \leq \dim(\mathbf{G}, \mathbf{X})$. We say that a Hecke-regular sequence $(Z_j, a_j, h_j)_{j=0}^m$ of $\text{Sh}_{\mathcal{K}}^{\text{tor}, n}$ is strata-based if for all j , $0 \leq j \leq m$, the reduced zero scheme of h_j is $\hat{S}_{\dim(\mathbf{G}, \mathbf{X}) - (j+1)}$. (Here we set $S_j = \hat{S}_j = \emptyset$ for all $j < 0$.)*

10.2. Going down, increasing the weight and going up. For the rest of §10.2, we make the assumptions of Theorem 3.4.1.

Given a Hecke regular subscheme $Z \subset \text{Sh}_{\mathcal{K}}^{\text{tor}, n}$, we write $\mathcal{H}_{\mathcal{K}}(Z_j; i, n, \eta)$ for the image of the Hecke algebra \mathcal{H} in $\text{End}(H^i(Z, \mathcal{V}_{\eta}^{\text{sub}}))$. If h is a global section of a line bundle on a scheme X , we write $Z(h)$ for the zero-scheme of h in X .

Lemma 10.2.1 (Going Down). *Suppose $(i, n, \eta, \mathcal{K})$ is a quadruple as in Theorem 3.4.1. Then there exists $b \in \mathbf{Z}_{\geq 1}$ and a strata-based, Hecke-regular sequence*

$$(10.2.1) \quad (Z_j, a_j, h_j)_{j=0}^i$$

such that one has the factorization

$$(10.2.2) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(Z_i; 0, n, \eta + b\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(i, n, \eta) & & \end{array}$$

Proof. By the assumptions of Theorem 3.4.1, the assumptions of Proposition 9.2.2 hold.

The proof is by induction on the length of the sequence (10.2.1). Thus assume that there exists a strata-based Hecke-regular sequence $\mathcal{Z}_J = (Z_j, a_j, h_j)_{j=0}^J$, an integer b_J and a factorization

$$(10.2.3) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(Z_J; i - J, n, \eta + b_J\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(i, n, \eta) & & \end{array}$$

Under this assumption, we shall prove that there exist $r, a'_{J+1}, b_{J+1} \in \mathbf{Z}_{\geq 1}$ and an injective section $h'_{J+1} \in H^0(Z(h_J^r), \omega^{a_{J+1}})$ such that

1. The sequence $\mathcal{Z}_{J+1} = (Z'_j, a'_j, h'_j)_{j=0}^{J+1}$ of length $J+1$, defined by $(Z_j, a_j, h_j)_{j=0}^{J-1} = (Z_j, a_j, h_j)_{j=0}^{J-1}$, followed by $(Z'_J, a'_J, h'_J) = (Z_J, ra_J, h_J^r)$ and ending with $(Z'_{J+1}, a'_{J+1}, h'_{J+1}) = (Z(h_J^r), a'_{J+1}, h'_{J+1})$, is a strata-based Hecke-regular sequence.
2. One has the factorization

$$(10.2.4) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(Z'_{J+1}; i - (J+1), n, \eta + b_{J+1}\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(i, n, \eta) & & \end{array}$$

The factorization (10.2.3) implies that $H^{i-J}(Z_J, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J}) \neq 0$. Moreover, it follows from Definition 7.2.3 that $\dim Z_J = \dim(\mathbf{G}, \mathbf{X}) - J$. By Lemma 7.3.1, there exists $r \in \mathbf{Z}_{\geq 1}$ such that

$$(10.2.5) \quad H^{i-J}(Z_J, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J + ra_J}) = 0.$$

Since the section $h_J \in H^0(Z_J, \omega^{a_J})$ is injective and Hecke-equivariant, it gives rise to a Hecke-equivariant short exact sequence of sheaves on Z_J :

$$(10.2.6) \quad 0 \rightarrow \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J} \rightarrow \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J + ra_J} \rightarrow (\mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J + ra_J})|_{Z(h_J^r)} \rightarrow 0.$$

The associated long exact sequence in cohomology gives

$$(10.2.7) \quad H^{i-(J+1)}(Z(h_J^r), \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J + ra_J}) \rightarrow H^{i-J}(Z_J, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J}) \rightarrow H^{i-J}(Z_J, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b_J + ra_J}).$$

Combining (10.2.5), (10.2.7) and Lemma 10.1.1, 2., yields the factorization (10.2.4). Moreover, since the sequence \mathcal{Z}_J is strata-based, one has $Z(h_J^r)_{\text{red}} = \hat{S}_{\dim(\mathbf{G}, \mathbf{X}) - (J+1)}$. Applying Theorems 3.1.3 and 8.1.1 gives $a'_{J+1} \in \mathbf{Z}_{\geq 1}$ and a section $h'_{J+1} \in H^0(Z(h_J^r), \omega^{a'_{J+1}})$ such that $Z(h'_{J+1})_{\text{red}} = \hat{S}_{\dim(\mathbf{G}, \mathbf{X}) - (J+2)}$. This completes the induction argument.

□

For the rest of §10.2, we shall use the notation and the strata-based Hecke-regular sequence constructed in Lemma 10.2.1.

Lemma 10.2.2 (Increasing the weight). *For all sufficiently large $r \in \mathbf{Z}_{\geq 1}$, one has the factorization*

$$(10.2.8) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(Z_i; 0, n, \eta + (b + ra_i)\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(Z_i; 0, n, \eta + b\eta_{\omega}) & & \end{array}$$

and for all j , $0 \leq j \leq i-1$, one has

$$(10.2.9) \quad H^1(Z_j, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b+ra_i-a_j}) = 0$$

Proof. Consider the injective, Hecke-equivariant section h_i pertaining to the strata-based, Hecke-regular sequence constructed in Lemma 10.2.1. For all $r \in \mathbf{Z}_{\geq 1}$, multiplication by h_i^r induces a Hecke-equivariant injection

$$(10.2.10) \quad H^0(Z_i, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^b) \hookrightarrow H^0(Z_i, \mathcal{V}_{\eta}^{\text{sub}} \otimes \omega^{b+ra_i}).$$

Applying Lemma 10.1.1, 1. gives the factorization (10.2.8). That the factorization satisfies the vanishing (10.2.9) for sufficiently large r follows from Lemma 7.3.1. □

Lemma 10.2.3 (Going up). *Suppose r satisfies (10.2.8) and (10.2.9). Then one has the factorization*

$$(10.2.11) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(0, n, \eta + (b + ra_i)\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(Z_i; 0, n, \eta + (b + ra_i)\eta_{\omega}) & & \end{array}$$

Proof. The proof is by downward induction, using the strata-based Hecke-regular sequence $(Z_j, a_j, h_j)_{j=0}^i$. Let $J \leq i$. For $J = i$ there is nothing to prove. Thus we assume that we have the factorization

$$(10.2.12) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(Z_J; 0, n, \eta + (b + ra_i)\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(Z_i; 0, n, \eta + (b + ra_i)\eta_{\omega}) & & \end{array}$$

and we prove that one also has the factorization

$$(10.2.13) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(Z_{J-1}; 0, n, \eta + (b + ra_i)\eta_{\omega}) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(Z_J; 0, n, \eta + (b + ra_i)\eta_{\omega}) & & \end{array}$$

Combining the factorizations (10.2.12) and (10.2.13) will then complete the induction.

We consider the short exact sequence of sheaves on Z_{J-1} given by multiplication by the injective, Hecke-equivariant section h_{J-1} :

$$(10.2.14) \quad 0 \rightarrow \mathcal{V}_\eta^{\text{sub}} \otimes \omega^{b+ra_i-a_{J-1}} \rightarrow \mathcal{V}_\eta^{\text{sub}} \otimes \omega^{b+ra_i} \rightarrow (\mathcal{V}_\eta^{\text{sub}} \otimes \omega^{b+ra_i})|_{Z_J} \rightarrow 0.$$

The long exact sequence gives

$$(10.2.15) \quad H^0(Z_{J-1}, \mathcal{V}_\eta^{\text{sub}} \otimes \omega^{b+ra_i}) \rightarrow H^0(Z_J, \mathcal{V}_\eta^{\text{sub}} \otimes \omega^{b+ra_i}) \rightarrow H^1(Z_{J-1}, \mathcal{V}_\eta^{\text{sub}} \otimes \omega^{b+ra_i-a_{J-1}}).$$

By (10.2.9), the right-most term in (10.2.15) is zero (given our choice of r). Hence Lemma 10.1.1, 2. gives (10.2.13). \square

Proof of Theorem 3.4.1: Apply Lemmas 10.2.1, Lemma 10.2.2 and Lemma 10.2.3 successively. \square

10.3. Increased regularity via the flag space.

10.3.1. *The flag space.* For every $n \in \mathbf{Z}_{\geq 1}$, the quotient of the P^0 -torsor \mathfrak{P}^0 on $\text{Sh}_\mathcal{K}^{\text{tor},0}$ by the Borel subgroup $B^{0,-}$ is represented by a smooth, projective $\text{Sh}_\mathcal{K}^{\text{tor},0}$ -scheme which we denote

$$(10.3.1) \quad \pi_0^{\text{Fl,Sh}} : \text{Fl}_\mathcal{K}^{\text{tor},0} \longrightarrow \text{Sh}_\mathcal{K}^{\text{tor},0}.$$

We call $\text{Fl}_\mathcal{K}^{\text{tor},0}$ the flag space of the Shimura datum (\mathbf{G}, \mathbf{X}) . For any scheme $S \rightarrow \text{Sh}_\mathcal{K}^{\text{tor},0}$, we have a bijection:

$$(10.3.2) \quad \text{Hom}_{\text{Sh}_\mathcal{K}^{\text{tor},0}}(S, \text{Fl}_\mathcal{K}^{\text{tor},0}) = \{B^{0,-}\text{-torsors on } S \text{ contained in } \mathfrak{P}^0 \times_{\text{Sh}_\mathcal{K}^{\text{tor},0}} S\}$$

From this description, we deduce immediately that the special fiber of $\text{Fl}_\mathcal{K}^{\text{tor},0}$ is the fiber product of $\text{Sh}_\mathcal{K}^{\text{tor},0}$ and $G\text{-ZipFlag}^\mu$ over $G\text{-Zip}^\mu$:

$$(10.3.3) \quad \text{Fl}_\mathcal{K}^{\text{tor},1} \simeq \text{Sh}_\mathcal{K}^{\text{tor},1} \times_{G\text{-Zip}^\mu} G\text{-ZipFlag}^\mu$$

By construction, the tower $(\text{Fl}_\mathcal{K}^{\text{tor},0})_{\mathcal{K}^p \subset \mathbf{G}(\mathbf{A}_f^p)}$ admits an action of $\mathbf{G}(\mathbf{A}_f^p)$ and the maps $\pi_0^{\text{Fl,Sh}}$ for $\mathcal{K}^p \subset \mathbf{G}(\mathbf{A}_f^p)$ give a map of towers with $\mathbf{G}(\mathbf{A}_f^p)$ -action.

By yet another application of the “associated sheaves” construction (*cf.* [Jan03, §5.8]), to every $\eta \in X^*(\mathbf{T}_\mathbf{C})$ there is associated in a functorial manner a Hecke-equivariant line bundle \mathcal{L}_η on $\text{Fl}_\mathcal{K}^{\text{tor},0}$, whose restriction to a fiber is the line bundle previously called \mathcal{L}_η in §2.1.5. As a result, if $\eta \in X_{+,c}^*(\mathbf{T}_\mathbf{C})$, then one has a canonical, Hecke-equivariant identification $\pi_{0,*}^{\text{Fl,Sh}}(\mathcal{L}_\eta) = \mathcal{V}_\eta$.

We define $\mathcal{L}_\eta^{\text{sub}}$ to be $\mathcal{L}_\eta \otimes (\pi_0^{\text{Fl,Sh}})^* \mathcal{O}(-D_\mathcal{K})$. Since $\mathcal{O}(-D_\mathcal{K})$ is a line bundle, the projection formula shows that the previous identification gives $\pi_{0,*}^{\text{Fl,Sh}}(\mathcal{L}_\eta^{\text{sub}}) = \mathcal{V}_\eta^{\text{sub}}$.

Remark 10.3.1. The \mathcal{O}_p -scheme $\text{Fl}_\mathcal{K}^{\text{tor},0}$ is a simultaneous generalization of aspects of work of Griffiths-Schmid [GS69] (see also Carayol [Car98]) on the one hand and of Ekedahl-van der Geer [EvdG09] on the other. We were inspired by both of these works.

1. Griffiths-Schmid studied homogenous complex manifolds of the form $\Gamma \backslash G/T$, where G is a connected, semisimple, real Lie group, T is a compact Cartan subgroup, G/T is endowed with a complex structure and Γ is an arithmetic subgroup. Carayol has termed these Griffiths-Schmid manifolds (*loc. cit.*). The complex manifold given by the complex points of $\text{Fl}_\mathcal{K}^{\text{tor},0}$ is an adelic

version of an “algebraic Griffiths-Schmid manifold” –one where the complex structure on G/T is such that G/T fibers holomorphically over a Hermitian symmetric domain (=a connected component of \mathbf{X}). Thus our construction gives a moduli interpretation of algebraic Griffiths-Schmid manifolds. We note that, for a different choice of complex structure on G/T (the “non-classical case”), the associated Griffiths-Schmid manifold has been shown to be not algebraic [GRT14].

2. In the Siegel case, Ekedahl-van der Geer defined and studied a flag space over \mathbf{F}_p by more elementary means than ours, namely in terms of full symplectic flags. Our construction offers a group-theoretic reinterpretation of their construction, which allows us to generalize to the general Hodge case.

10.3.2. *Proof of Corollary 3.4.2.* There are two ingredients needed to deduce Corollary 3.4.2. The first is a “factorization of Hecke algebras” lemma on the flag space. The second is to compare cohomology on the flag space and on the Shimura variety by means of Kempf’s vanishing theorem.

Our factorization lemma on the flag space is:

Lemma 10.3.2. *Suppose $\chi \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ is in the image of $X_+^*(\mathbf{T}_{\mathbf{C}})$ under $h_{w_0,M}$ (defined by (6.1.3) for $w = w_{0,M}$). Then for all $\eta' \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ one has the factorization*

$$(10.3.4) \quad \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H}_{\mathcal{K}}(0, n, \eta' + \chi) \\ \downarrow & \swarrow & \\ \mathcal{H}_{\mathcal{K}}(0, n, \eta') & & \end{array}$$

Proof. By Theorem 5.2.2, the hypothesis on χ implies that there exists a Hecke-equivariant, injective section $h_{\chi} \in H^0(\mathrm{Fl}_{\mathcal{K}}^{\mathrm{tor},1}, \mathcal{L}_{\chi})$. At this point, the argument is analogous to what was done repeatedly in §10.2, so we summarize it in one sentence: Considering the first map in the long exact sequence associated to the short exact sequence obtained by multiplication by h_{χ} and applying Lemma 10.1.1, part 1. yields the desired factorization. \square

Comparison of cohomology on $\mathrm{Fl}_{\mathcal{K}}^{\mathrm{tor},n}$ and $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}$ is achieved by the following:

Lemma 10.3.3. *Suppose $\eta' \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$. Then for all $n \in \mathbf{Z}_{\geq 0}$ and all $i \in \mathbf{Z}_{\geq 1}$, one has*

$$(10.3.5) \quad R^i \pi_{n,*}^{\mathrm{Fl},\mathrm{Sh}} \mathcal{L}_{\eta'}^{\mathrm{sub}} = 0.$$

Consequently we have a Hecke-equivariant isomorphism

$$(10.3.6) \quad H^i(\mathrm{Fl}_{\mathcal{K}}^{\mathrm{tor},n}, \mathcal{L}_{\eta'}^{\mathrm{sub}}) \cong H^i(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},n}, \mathcal{V}_{\eta'}^{\mathrm{sub}})$$

for all $i \in \mathbf{Z}_{\geq 0}$.

Proof. By Kempf’s vanishing theorem [Jan03, II, Chapter 4], the dominance assumption $\eta' \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$ implies that, for every point $x \in \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ and all $i \in \mathbf{Z}_{\geq 1}$, one has $H^i((\mathrm{Fl}_{\mathcal{K}}^{\mathrm{tor},1})_x, \mathcal{L}_{\eta'}) = 0$. Applying Lemma 7.1.1 to the local ring $\mathcal{O}_{\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0},x}$ proves that all the stalks of $R^i \pi_{n,*}^{\mathrm{Fl},\mathrm{Sh}} \mathcal{L}_{\eta'}^{\mathrm{can}}$ are zero, so the sheaf is zero. It remains

to show the same with “can” replaced by “sub”, and this follows from the projection formula. \square

Proof of Corollary 3.4.2: Apply Lemmas 10.3.2 and 10.3.3 with $\eta' = \eta + a\eta_\omega$. \square

11. APPLICATIONS AND OTHER RESULTS

11.1. Galois representations. We shall now explain how to deduce the results of §3.5 from our factorization results (Theorem 3.4.1 and its Corollary 3.4.2). The proofs in the case of general groups (§3.5.1) are analogues to the case of unitary groups (§3.5.2), except that in the former case we assume Condition 2.4.2 and in the latter case we use Corollary 1.3 of [HLTT13]. (We note that Corollary 1.3 of [HLTT13] is a concise formulation of combining Shin’s results [Shi11, Theorem 1.2] and [Gol14, Theorem A.1], which build on the work of many people, see also Remark 11.1.1 below.) Moreover, the argument we use is analogous to the one introduced by Taylor [Tay91], and then applied in [Jar97], [Gol14] and [HLTT13]. For these reasons, we only treat the case of general groups (§3.5.1).

We begin with the case of torsion.

Proof of Theorem 3.5.1, Case [Tor]. Fix $(r; i, n, \eta, \mathcal{K})$ as in the statement of the theorem. Choose $\chi_\eta \in X^*(\mathbf{T}_{\mathbf{C}})$ in the image of $X_+^*(\mathbf{T}_{\mathbf{C}})$ under $h_{w_{0,M}}$, satisfying (1) and (2) of Condition 2.4.2 and such that χ_η is Φ_c^\vee -regular. For all $k \in \mathbf{Z}_{\geq 1}$, there exists $a_0 = a_0(k) \in \mathbf{Z}_{\geq 1}$ such that $-w_{0,c}(\eta + a\eta_\omega + k\chi_\eta + \rho) - \rho$ is Δ^\vee -dominant and Φ^\vee -regular for all $a \geq a_0$. Suppose

$$(11.1.1) \quad f \in H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}, \mathcal{V}_{\eta+a\eta_\omega+k\chi_\eta}^{\mathrm{sub}})$$

is a Hecke eigenform and let $\pi(f)$ be the cuspidal automorphic representation of \mathbf{G} that it generates. By the Casselman-Osbourne Theorem (cf. [Har88, Proposition 3.1.4]), the infinitesimal character of $\pi(f)$ is $-w_{0,c}(\eta + a\eta_\omega + k\chi_\eta + \rho)$. By Salamanca-Riba’s theorem (§2.2.2), the archimedean component $\pi(f)_\infty$ is discrete series.

To alleviate notation, write $\mathcal{H}_{a,k}^n = \mathcal{H}_{\mathcal{K}}(0, n, \eta + a\eta_\omega + k\chi_\eta)$ for all $n \in \mathbf{Z}_{\geq 0}$. Applying 2. of Condition 2.4.2 to all eigenforms f as in (11.1.1) (with (i, η, \mathcal{K}) fixed), yields a unique semisimple Galois representation

$$(11.1.2) \quad \rho_{a,k} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow GL(m, \mathcal{H}_{a,k}^0 \otimes \overline{\mathbf{Q}}_p)$$

such that $(\mathrm{tr} \rho_{a,k})(\mathrm{Frob}_v^j) = T_v^{(j)}(r; i, n, \eta, \mathcal{K})$ for all $j \in \mathbf{Z}_{\geq 1}$ (see §2.3.2). Let

$$(11.1.3) \quad \rho_{a,k}^{\mathrm{pseudo}} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathcal{H}_{a,k}^0$$

be the induced pseudo-representation.

By Corollary 7.1.3 and Lemma 10.1.1, if a is sufficiently large in terms of k and η , then we have the factorization

$$(11.1.4) \quad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{H}_{a,k}^0 \\ & \searrow & \downarrow \\ & & \mathcal{H}_{a,k}^n \end{array}$$

For sufficiently large a , Corollary 3.4.2 (applied with $\chi = k\chi_\eta$) gives a second factorization, namely (3.4.2). Composing the two factorizations gives a third one:

$$(11.1.5) \quad \begin{array}{ccc} \mathcal{H} & \xrightarrow{\quad} & \mathcal{H}_{a,k}^0 \\ & \searrow & \downarrow \\ & & \mathcal{H}_K(i, n, \eta) \end{array} .$$

Composing the vertical map in (11.1.5) with the pseudo-representation $\rho_{a,k}^{\text{pseudo}}$ gives a pseudo-representation $R_{p,\iota}(r; i, n, \eta, K)$ satisfying (3.5.2). Uniqueness follows from the Tchebotarev density theorem. \square

Next we turn to the case of non-degenerate limits.

Proof of Theorem 3.5.1, Case [LDS]. Let π be as in the statement of the theorem. Since π is unramified at p , there exists a p -hyperspecial level K such that $\pi^K \neq 0$. Let $\gamma_\pi \in \pi^K$ be a Hecke eigenform (for \mathcal{H}). By (2.2.7), we have that γ_π is a Hecke eigenclass in $H^i(\text{Sh}_K^{\text{tor},0} \otimes \overline{\mathbf{Q}_p}, \mathcal{V}_\eta^{\text{sub}})$, where i and η are given by Theorem 2.2.1. After replacing \mathcal{O}_p by a finite extension, we may assume that γ_π lies in $H^i(\text{Sh}_K^{\text{tor},0}, \mathcal{V}_\eta^{\text{sub}})$. Reduction modulo \mathfrak{p}^n gives Hecke eigenclasses $\gamma_\pi^n \in H^i(\text{Sh}_K^{\text{tor},0}, \mathcal{V}_\eta^{\text{sub}})$.

Let

$$(11.1.6) \quad \theta_n : \mathcal{H}_K(i, n, \eta) \longrightarrow \mathcal{O}_p/\mathfrak{p}^n$$

be the eigenvalue map of γ_π^n . The compositions $\theta_n \circ R_{p,\iota}(r; i, n, \eta, K)$ form a \mathfrak{p} -adic system of pseudo-representations. Hence their limit gives a $\overline{\mathbf{Q}_p}$ -valued pseudo-representation. By [Tay91, Theorem 1], this $\overline{\mathbf{Q}_p}$ -valued pseudo-representation is the trace of a unique true, semisimple representation satisfying the desiderata of Theorem 3.5.1. Uniqueness follows again from the Tchebotarev density theorem. \square

Remark 11.1.1. In view of the “change of weight” afforded by Corollary 3.4.2, in the case of unitary similitude groups, we only use a weak version of Corollary 1.3 of [HLTT13], where the archimedean component satisfies the type of regularity in Condition 2.4.2. Therefore our results use neither Shin’s base change [Gol14, Theorem A.1], nor the work of Chenevier-Harris [CH13]; rather it suffices for us to combine [Shi11, Theorem 1.2] with Labesse’s more restricted base change¹⁶ [Lab11].

11.2. Proof of affineness, Part (2). In this section, we prove part (2) of Theorem 3.3.1, concerning extended Ekedahl-Oort strata in the minimal compactification.

Recall that, for $w \in {}^I W$, we denote by S_w (resp. S_w^{tor}) the Ekedahl-Oort stratum in Sh_K^1 (resp. $\text{Sh}_K^{\text{tor},1}$) corresponding to w . We define naturally a subset S_w^{min} in $\text{Sh}_K^{\text{min},1}$ by:

$$(11.2.1) \quad S_w^{\text{min}} := \pi_1^{\text{tor},\text{min}}(S_w)$$

¹⁶To be completely precise, Labesse’s result has the disadvantage of being stated for unitary groups rather than unitary similitude groups, and to assume $F^+ \neq \mathbf{Q}$, so in that respect we do use Shin’s [Gol14, Theorem A.1], but not concerning the regularity of the archimedean component.

We can thus write $\mathrm{Sh}_{\mathcal{K}}^{\min,1}$ as a union:

$$(11.2.2) \quad \mathrm{Sh}_{\mathcal{K}}^{\min,1} = \bigcup_{w \in {}^I W} S_w^{\min}$$

Remark 11.2.1. In the case of Siegel-type Shimura varieties, except for the statement concerning affineness, Theorem 3.3.1 was proved by Ekedahl-Van der Geer¹⁷ in [EvdG09], Lemma 5.1 (see particularly the last sentence of §5). Furthermore, the authors point out that it is thus equivalent to (i) define the Ekedahl-Oort strata on $\mathrm{Sh}_{\mathcal{K}}^{\min,1}$ as the images of strata in $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ (as above), and (ii) define a closed stratum in $\mathrm{Sh}_{\mathcal{K}}^{\min,1}$ as the closure of a stratum S_w in $\mathrm{Sh}_{\mathcal{K}}^{\min,1}$ and then define a stratum as the complement of the smaller closed strata.

Proof of Theorem 3.3.1, Part (2): We will deduce Theorem 3.3.1 by combining our results (Theorem 3.1.3, Corollary 6.3.2) with the results of Ekedahl-van der Geer in the Siegel case described in Remark 11.2.1.

Recall that we have an embedding of Shimura data $(\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(2g), \mathbf{X}_g)$ into a Siegel Shimura datum. This embedding induces a finite morphism $\iota : \mathrm{Sh}_{\mathcal{K}}^1 \rightarrow \mathcal{A}_{\mathcal{K}'}$, where we denote by $\mathcal{A}_{\mathcal{K}'}$ the base change to $k = \overline{\mathbf{F}}_p$ of the special fiber of the Siegel-type Shimura variety attached to $(GSp(2g), \mathbf{X}_g)$. Also, we consider $\mathrm{Sh}_{\mathcal{K}}^1$ as a k -scheme (we omit the notation for base-change). Denote by $\mu' : \mathbf{G}_{m,k} \rightarrow GSp(2g)$ the cocharacter $\iota \circ \mu$. We assume that we have toroidal compactifications $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ and $\mathcal{A}_{\mathcal{K}'}^{\mathrm{tor}}$, a finite morphism $\iota^{\mathrm{tor}} : \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1} \rightarrow \mathcal{A}_{\mathcal{K}'}^{\mathrm{tor}}$ and a finite morphism $\iota^{\min} : \mathrm{Sh}_{\mathcal{K}}^{\min,1} \rightarrow \mathcal{A}_{\mathcal{K}'}^{\min}$ such that we have a commutative diagram:

$$\begin{array}{ccc}
 G\text{-}\mathrm{Zip}^{\mu} & \xrightarrow{\iota^{\mathrm{Zip}}} & GSp(2g)\text{-}\mathrm{Zip}^{\mu'} \\
 \uparrow \zeta^{\mathrm{tor}} & & \uparrow \zeta_{\mathcal{A}}^{\mathrm{tor}} \\
 \mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1} & \xrightarrow{\iota^{\mathrm{tor}}} & \mathcal{A}_{\mathcal{K}'}^{\mathrm{tor}} \\
 \uparrow \pi_1^{\mathrm{tor},\min} & & \downarrow \pi_{1,\mathcal{A}}^{\mathrm{tor},\min} \\
 \mathrm{Sh}_{\mathcal{K}}^1 & \xrightarrow{\iota} & \mathcal{A}_{\mathcal{K}'} \\
 \downarrow & & \downarrow \\
 \mathrm{Sh}_{\mathcal{K}}^{\min,1} & \xrightarrow{\iota^{\min}} & \mathcal{A}_{\mathcal{K}'}^{\min}
 \end{array}$$

Let $x \rightarrow \mathrm{Sh}_{\mathcal{K}}^{\min,1}$ be a geometric point, and let $X := (\pi_1^{\mathrm{tor},\min})^{-1}(x)$ be its fiber in $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$, which is connected. Since the S_w^{\min} are disjoint in the case of $\mathcal{A}_{\mathcal{K}'}$, the commutativity of the diagram implies that $\iota^{\mathrm{tor}}(X)$ is contained in a single Ekedahl-Oort stratum of $\mathcal{A}_{\mathcal{K}'}^{\mathrm{tor}}$. Hence $\zeta^{\mathrm{tor}}(X)$ is a connected subset of $G\text{-}\mathrm{Zip}^{\mu}$ that maps to a single point by ι^{Zip} . Since ι^{Zip} has discrete fibers by Theorem 6.3.1, we deduce that $\zeta^{\mathrm{tor}}(X)$ is a singleton, hence X is contained in a single Ekedahl-Oort stratum. This shows:

$$(11.2.3) \quad (\pi_1^{\mathrm{tor},\min})^{-1}(S_w^{\min}) = S_w^{\mathrm{tor}}.$$

¹⁷Moreover, the authors prove for the Siegel case that each stratum S_w is dense in S_w^{tor} , and that the S_w^{tor} form a stratification of $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ (i.e the closure of a stratum is a union of strata). In the general Hodge-type case, these properties would follow from the smoothness of the map ζ^{tor} .

This shows that the S_w^{\min} are pairwise disjoint.

Now we prove that S_w^{\min} is locally closed in $\mathrm{Sh}_{\mathcal{K}}^{\min,1}$. Let $S_{\mathcal{A},w}^{\min}$ be the Siegel Ekedahl-Oort stratum in \mathcal{A}^{\min} containing $\iota^{\min}(S_w^{\min})$, and denote by $S_{\mathcal{A},w}^{\mathrm{tor}}$ the corresponding Ekedahl-Oort stratum in $\mathcal{A}_{\mathcal{K}'}^{\mathrm{tor}}$. Define:

$$(11.2.4) \quad Z^{\min} := (\iota^{\min})^{-1}(S_{\mathcal{A},w}^{\min})$$

$$(11.2.5) \quad Z^{\mathrm{tor}} := (\iota^{\mathrm{tor}})^{-1}(S_{\mathcal{A},w}^{\mathrm{tor}})$$

it is clear these sets are locally closed and that one has $(\pi_1^{\mathrm{tor},\min})^{-1}(Z^{\min}) = Z^{\mathrm{tor}}$. Hence the map $\pi_1^{\mathrm{tor},\min} : Z^{\mathrm{tor}} \rightarrow Z^{\min}$ is closed. Since S_w^{tor} is closed in Z^{tor} , we deduce that S_w^{\min} is closed in Z^{\min} , hence is locally closed in $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$.

Finally, we prove that S_w^{\min} (endowed with the reduced subscheme structure) is affine. denote by \hat{S}_w^{tor} the closed Ekedahl-Oort stratum corresponding to w (i.e the preimage by ζ^{tor} of the closure of $\{w\}$ in $G\text{-Zip}^{\mu}$). By pull-back from $G\text{-Zip}^{\mu}$, we obtain by Theorem 3.1.2 a section

$$(11.2.6) \quad H_w^{\mathrm{tor}} \in H^0(\hat{S}_w^{\mathrm{tor}}, \omega^N)$$

for some $N \geq 1$, whose non-vanishing locus is S_w^{tor} . Set $A_w := \pi_1^{\mathrm{tor},\min}(\hat{S}_w^{\mathrm{tor}})$, and denote by π_w the map $\hat{S}_w^{\mathrm{tor}} \rightarrow A_w$ induced by $\pi_1^{\mathrm{tor},\min}$. Using Equation (11.2.3), we see readily that π' has connected fibers, so we deduce $\pi'_*(\mathcal{O}_{\hat{S}_w^{\mathrm{tor}}}) = \mathcal{O}_{A_w}$. The projection formula shows that the same holds for the line bundle ω . Hence the section H_w^{tor} descends to a section H_w^{\min} over A_w . Moreover, the non-vanishing locus of H_w^{\min} is S_w^{\min} . Since ω is ample on $\mathrm{Sh}_{\mathcal{K}}^{\min,1}$, it follows that S_w^{\min} is affine. \square

11.3. Systems of Hecke eigenvalues over S_e . Recall that it is now known that the stratum S_e is nonempty and zero-dimensional.

Proof of Theorem 3.6.1. Large parts of our argument are identical to those of Reduzzi [Red03], which in turn are identical to those of Ghitza [Ghi04], so we shall be brief. The argument has three steps, which are somewhat analogous to the steps “Going down”, “Increasing the weight” and “Going up” in the proof of Theorem 3.4.1. The first step is to show that every system of Hecke eigenvalues that appears in (3.6.3) appears in (3.6.4). The second step is to show that there exists $m \in \mathbf{Z}_{\geq 1}$, depending only on (\mathbf{G}, \mathbf{X}) , such that, for all η , the systems appearing in $H^i(S_e, \mathcal{V}_{\eta})$ are the same as those appearing in $H^i(S_e, \mathcal{V}_{\eta} \otimes \omega^m)$. The final step then shows that, if m' is a sufficiently large multiple of m , then every system appearing in $H^i(S_e, \mathcal{V}_{\eta} \otimes \omega^{m'})$ also appears in $H^i(\mathrm{Sh}_{\mathcal{K}}^1, \mathcal{V}_{\eta} \otimes \omega^{m'})$.

Step 1: Let $f \in H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}, \mathcal{V}_{\eta})$ be a Hecke eigenform. Let \mathcal{I} be the ideal sheaf of S_e in $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$. Then we have a Hecke-equivariant short exact sequence

$$(11.3.1) \quad 0 \rightarrow \mathcal{I} \otimes \mathcal{V}_{\eta} \rightarrow \mathcal{V}_{\eta} \rightarrow \mathcal{V}_{\eta}|_{S_e} \rightarrow 0.$$

Since $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},0}$ is Noetherian and S_e is nonempty, there exists a largest $j \in \mathbf{Z}_{\geq 0}$ such that

$$f \in H^0(\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}, \mathcal{I}^j \otimes \mathcal{V}_{\eta}).$$

Then the image \bar{f} of f in $H^0(\mathcal{I}^j/\mathcal{I}^{j+1} \otimes \mathcal{V}_{\eta})$ is non-zero. Now we have $\mathcal{I}^j/\mathcal{I}^{j+1} = \mathrm{Sym}^j(\mathcal{I}/\mathcal{I}^2)$. Moreover, since S_e and $\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}$ are smooth, and the former is zero-dimensional, a standard exact sequence of differentials simplifies to an isomorphism $\mathcal{I}/\mathcal{I}^2 \cong (\Omega_{\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}}^1)|_{S_e}$ of sheaves on S_e . Hence we have $\bar{f} \in H^0(S_e, \mathrm{Sym}^j(\Omega_{\mathrm{Sh}_{\mathcal{K}}^{\mathrm{tor},1}}^1))$.

So the system of Hecke eigenvalues of f also appears in $H^0(S_e, \text{Sym}^j(\Omega_{\text{Sh}_{\mathcal{K}}^{\text{tor},1}}^1))$. But now the Kodaira-Spencer isomorphism (see [Mad, 5.3.7] for the general Hodge case) and the functoriality of the “associated sheaves” construction imply that in fact this system of Hecke eigenvalues appears in $H^0(S_e, \mathcal{V}_{\eta'})$ for some $\eta' \in X_{+,c}^*(\mathbf{T}_{\mathbf{C}})$.

Step 2: Let $f' \in H^0(S_e, \mathcal{V}_{\eta})$ be a Hecke eigenform. By Theorem 3.1.2, there exists a nowhere vanishing, Hecke-equivariant section $f_e \in H^0(S_e, \omega^m)$ for some m . Then for all positive multiples m' of m , multiplication by $f_e^{m'/m}$ gives a Hecke-equivariant isomorphism

$$H^0(S_e, \mathcal{V}_{\eta}) \xrightarrow{\sim} H^0(S_e, \mathcal{V}_{\eta} \otimes \omega^{m'}).$$

Therefore the system of Hecke eigenvalues of f' also appears in $H^0(S_e, \mathcal{V}_{\eta} \otimes \omega^{m'})$.

Step 3: Since ω_{\min} is ample on $\text{Sh}_{\mathcal{K}}^{\min,1}$, and since $\mathcal{O}(-D_{\mathcal{K}})$ is relatively ample, there exists $k \in \mathbf{Z}_{\geq 1}$ such that $\omega^k(-D_{\mathcal{K}})$ is ample on $\text{Sh}_{\mathcal{K}}^{\text{tor},1}$. Since $S_e \cap D_{\mathcal{K}} = \emptyset$, we have $\omega^k(-D_{\mathcal{K}})|_{S_e} = \omega^k|_{S_e}$. By Serre vanishing, if m' is large enough and divisible by k , then

$$H^1(\text{Sh}_{\mathcal{K}}^{\text{tor},1}, \mathcal{V}_{\eta} \otimes \mathcal{I} \otimes \omega^{m'}(-\frac{m'}{k}D_{\mathcal{K}})) = 0.$$

But twisting (11.3.1) by $\omega^{m'}(-m'/kD_{\mathcal{K}})$ and passing to cohomology, the vanishing of H^1 above gives a surjection

$$(11.3.2) \quad H^0(\text{Sh}_{\mathcal{K}}^{\text{tor},1}, \mathcal{V}_{\eta} \otimes \omega^{m'}(-\frac{m'}{k}D_{\mathcal{K}})) \twoheadrightarrow H^0(S_e, \mathcal{V}_{\eta} \otimes \omega^{m'}).$$

Therefore the system of Hecke eigenvalues of f' also appears in $H^0(\text{Sh}_{\mathcal{K}}^{\text{tor},1}, \mathcal{V}_{\eta} \otimes \omega^{m'}(-\frac{m'}{k}D_{\mathcal{K}}))$. Since the latter is Hecke-equivariantly contained in $H^0(\text{Sh}_{\mathcal{K}}^{\text{tor},1}, \mathcal{V}_{\eta} \otimes \omega^{m'})$, the system of f' appears in $H^0(\text{Sh}_{\mathcal{K}}^{\text{tor},1}, \mathcal{V}_{\eta} \otimes \omega^{m'})$ as well.

This completes the proof that the system of Hecke eigenvalues which appear in (3.6.3) and (3.6.4) are the same. The finiteness statement is then a trivial consequence, as was observed in [Ser96], [Ghi04], [Red03]. \square

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APPENDIX A. THE HODGE LINE BUNDLE IS QUASI-CONSTANT

We now prove Theorem 2.1.3, that the Hodge line bundle (§2.1.6) is quasi-constant on Weyl-Galois orbits. The proof relies heavily on Deligne’s analysis of symplectic embeddings of Shimura data [Del77, §1.3]. Except for new notation introduced below, the notation pertaining to root data is that of §2.1.4. In particular, recall that Δ denotes the set of simple roots of $\mathbf{T}_{\mathbf{C}}$ in $\mathbf{G}_{\mathbf{C}}$ and that μ and Δ are

chosen compatibly so that μ is Δ -dominant and the pairing of any root with μ is equal to one if and only if the root is positive and noncompact.

Roughly, the proof breaks down into three steps, given in terms of the structure of the composition $\rho := \text{Std} \circ \varphi : \mathbf{G} \rightarrow GL(V)$: We first reduce to the case that ρ is irreducible over \mathbf{Q} (Lemma A.1) and then we reduce to a question about fundamental weights (Lemmas A.2- A.5). The question about fundamental weights is a simple computation which can be done case-by-base. The crux of the argument is the middle step, which requires a careful analysis of the restriction of ρ to \mathbf{L} .

Proof of Theorem 2.1.3. The composition $\rho := \text{Std} \circ \varphi$ is a representation of \mathbf{G} defined over \mathbf{Q} (since it is the composition of two morphisms which are both defined over \mathbf{Q}). The composition of $\rho_{\mathbf{R}} := \rho \otimes \mathbf{R}$ with $h : \mathbf{S} \rightarrow \mathbf{G}_{\mathbf{R}}$ yields a polarized \mathbf{R} -Hodge structure of type $\{(-1, 0), (0, -1)\}$ which we denote $h_{\varphi} : \mathbf{S} \rightarrow GL(V \otimes \mathbf{R})$. The pair (V, h_{φ}) is a polarized \mathbf{Q} -Hodge structure.

Lemma A.1. *In order to prove Theorem 2.1.3, it suffices to treat the case that the representation ρ is irreducible over \mathbf{Q} .*

Proof. Since \mathbf{Q} has characteristic zero and \mathbf{G} is connected and reductive, the category of \mathbf{Q} -algebraic representations of \mathbf{G} is semisimple. Suppose $\rho = \rho_1 \oplus \cdots \oplus \rho_m$, where for all i , $\rho_i : \mathbf{G} \rightarrow GL(V_i)$ is a \mathbf{Q} -irreducible representation of \mathbf{G} . The representation ρ_i induces a polarized \mathbf{Q} -Hodge structure on V_i , again of type $\{(-1, 0), (0, -1)\}$ (the restriction of the polarization on V to V_i is again a polarization). The polarization on V_i yields a non-degenerate alternating form ψ_i such that ρ_i factors through an embedding $\varphi_i : \mathbf{G} \hookrightarrow GSp(V, \psi_i)$. By construction this embedding of groups induces an embedding of Shimura data

$$\varphi : (\mathbf{G}, \mathbf{X}) \hookrightarrow (GSp(V, \psi_i), \text{Class}(\varphi_i \circ h)),$$

where $\text{Class}(\varphi_i \circ h)$ indicates the $GSp(V \otimes \mathbf{R}, \psi_i)$ -conjugacy class. It is straightforward that

$$\Omega(\varphi) = \bigoplus_{i=1}^m \Omega(\varphi_i) \text{ and } \omega(\varphi) = \bigotimes_{i=1}^m \omega(\varphi_i),$$

from which the reduction follows. \square

In view of Lemma A.1, we suppose for the rest of the proof of Theorem 2.1.3 that ρ is irreducible.

Let $\tilde{\mathbf{G}}$ be the semi-simple simply-connected cover of the derived subgroup \mathbf{G}^{der} of \mathbf{G} . We may identify Δ with the set of simple roots of $\tilde{\mathbf{T}}_{\mathbf{C}}$ in $\tilde{\mathbf{G}}_{\mathbf{C}}$ by means of the natural morphism of root data corresponding to $\tilde{\mathbf{G}}_{\mathbf{C}} \rightarrow \mathbf{G}_{\mathbf{C}}$.

By restriction to \mathbf{G}^{der} and composing with the projection $\tilde{\mathbf{G}} \twoheadrightarrow \mathbf{G}^{\text{der}}$ we get a representation $\tilde{\rho}$ of $\tilde{\mathbf{G}}$. Since ρ is irreducible, so is $\tilde{\rho}$ (cf. [Jan03, II, 2.10]). It follows that the irreducible factors of $\tilde{\rho}_{\mathbf{C}}$ are permuted transitively by $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and that every irreducible factor of $\tilde{\rho}_{\mathbf{C}}$ appears with the same multiplicity, namely the Schur index of $\tilde{\rho}$ (cf. [Ser77, §12.2]¹⁸)

Let $(G_i)_{1 \leq i \leq k}$ be the simple factors over \mathbf{C} of the adjoint group $\mathbf{G}_{\mathbf{C}}^{\text{ad}}$ and let \tilde{G}_i be the simply-connected covering of G_i . Since $\tilde{\mathbf{G}}_{\mathbf{C}}$ and \tilde{G}_i are simply-connected,

¹⁸The reference treats the case of finite groups, but as is well-known, the argument can be translated to the language of algebraic groups.

we have

$$(A.1) \quad \tilde{\mathbf{G}}_{\mathbf{C}} \cong \prod_{i=1}^k \tilde{G}_i.$$

Let ρ' be an irreducible factor of $\tilde{\rho}_{\mathbf{C}}$. By Deligne's classification of symplectic embeddings, specifically [Del77, 1.3.7], the highest weight of ρ' is a fundamental weight and so ρ' factors through \tilde{G}_i for some i . Thus we have a decomposition

$$(A.2) \quad \tilde{\rho}_{\mathbf{C}} = \bigoplus_{i=1}^k \rho_i,$$

where ρ_i is the direct sum of those ρ' that contribute to the factor \tilde{G}_i .

Since $\tilde{\rho}$ is \mathbf{Q} -irreducible, the above-cited result about fundamental weights implies that we may assume \mathbf{G}^{ad} is \mathbf{Q} -simple. Therefore $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ permutes the \tilde{G}_i transitively (in particular the G_i are pairwise isomorphic). If $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ conjugates \tilde{G}_i to \tilde{G}_j , then σ induces an isomorphism of $\sigma\rho_i$ with ρ_j . In particular, the multiplicity of any irreducible ρ' in ρ_i is independent of i .

The restriction of $\rho_{\mathbf{C}}$ to $\mathbf{L}_{\mathbf{C}}$ is equal to $V^{-1,0} \oplus V^{0,-1}$, the sum of the graded pieces¹⁹ of the \mathbf{R} -Hodge structure $(V_{\mathbf{R}}, \rho)$. The character $-\eta_{\omega}$ is the determinant of the $\mathbf{L}_{\mathbf{C}}$ -representation $V^{-1,0}$. Hence $-\eta_{\omega}$ is the sum of the $\mathbf{T}_{\mathbf{C}}$ -weights of $V^{-1,0}$ (counted with multiplicity).

Let $\tilde{\mathbf{L}}$ (resp. $\tilde{\mathbf{T}}$) be the preimage of $(\mathbf{L}/Z(\mathbf{G}))$ (resp. $\mathbf{T}/Z(\mathbf{G})$) in $\tilde{\mathbf{G}}$ under $\tilde{\mathbf{G}} \rightarrow \mathbf{G}^{\text{ad}}$. Let $\tilde{V}^{-1,0}$ (resp. $\tilde{V}^{0,-1}$) be the restriction of $V^{-1,0}$ (resp. $V^{0,-1}$) to $\tilde{\mathbf{L}}_{\mathbf{C}}$, so that we have an identification of $\tilde{\mathbf{L}}_{\mathbf{C}}$ -representations:

$$\text{Res}_{\tilde{\mathbf{L}}_{\mathbf{C}}}^{\tilde{\mathbf{G}}_{\mathbf{C}}} \tilde{\rho}_{\mathbf{C}} = \tilde{V}^{-1,0} \oplus \tilde{V}^{0,-1}$$

Let μ^{ad} be the composition of $\mu : \mathbf{G}_m \rightarrow \mathbf{G}_{\mathbf{C}}$ with the projection $\mathbf{G}_{\mathbf{C}} \twoheadrightarrow \mathbf{G}_{\mathbf{C}}^{\text{ad}}$. Since μ is minuscule, so is μ^{ad} . Let $\tilde{\mu}$ be the fractional lifting ("relèvement fractionnaire", [Del77, 1.3.4]) of μ^{ad} to $\tilde{\mathbf{G}}_{\mathbf{C}}$.

Let ρ' be an irreducible factor of $\tilde{\rho}_{\mathbf{C}}$. By Lemma 1.3.5 of *loc. cit.*, ρ' has two $\tilde{\mu}$ -weights given by a and $a+1$ for some $a \in \mathbf{Q}$. In other words, as ξ runs through the $\tilde{\mathbf{T}}_{\mathbf{C}}$ -weights of $\tilde{\rho}_{\mathbf{C}}$, the pairing $\langle \xi, \tilde{\mu} \rangle$ takes the two values a and $a+1$.

Lemma A.2. *Let ξ be a weight of ρ' . Then ξ is a weight of $\tilde{V}^{-1,0}$ (resp. $\tilde{V}^{0,-1}$) if and only if $\langle \xi, \tilde{\mu} \rangle = a+1$ (resp. $\langle \xi, \tilde{\mu} \rangle = a$).*

Proof. This follows easily from the proof of the aforementioned Lemma 1.3.5 of *loc. cit.* \square

Let \tilde{L}_i be the intersection of \tilde{G}_i with the centralizer, in $\tilde{\mathbf{G}}_{\mathbf{C}}$, of the fractional lifting $\tilde{\mu}$. Then for every i , either \tilde{L}_i is the Levi of a maximal parabolic of \tilde{G}_i , or $\tilde{L}_i = \tilde{G}_i$. For every i with $\tilde{L}_i \neq \tilde{G}_i$, let α_i be the unique simple root of \tilde{G}_i which is not a root of \tilde{L}_i .

Lemma A.3. *Let ρ' be an irreducible factor of $\tilde{\rho}_{\mathbf{C}}$ with highest weight ξ (relative to $(\tilde{\mathbf{G}}_{\mathbf{C}}, \tilde{\mathbf{T}}_{\mathbf{C}})$). Then ξ is a weight of $\tilde{V}^{-1,0}$.*

¹⁹Note that, in general, the two pieces $V^{-1,0}, V^{0,-1}$ are not irreducible as $\mathbf{L}_{\mathbf{C}}$ -representations. However, they are irreducible in the special case $(\mathbf{G}, \mathbf{X}) = (GSp(2g), \mathbf{X}_g)$.

Proof. Let a and $a + 1$ be the two $\tilde{\mu}$ -weights of ρ' . Since ρ' admits two distinct $\tilde{\mu}$ -weights, it admits a $\tilde{\mathbf{T}}_{\mathbf{C}}$ -weight ξ' whose pairing with $\tilde{\mu}$ is different from that of ξ with $\tilde{\mu}$. By the property characterizing the highest weight, $\xi - \xi'$ is a non-negative linear combination of simple roots. Since $\tilde{\mu}$ is Δ^\vee -dominant, $\langle \xi - \xi', \tilde{\mu} \rangle \geq 0$, but by our choice of ξ' , one has $\langle \xi - \xi', \tilde{\mu} \rangle \neq 0$. Hence $\langle \xi - \xi', \tilde{\mu} \rangle = 1$ and $\langle \xi, \tilde{\mu} \rangle = a + 1$. So ξ is a weight of $\tilde{V}^{-1,0}$ by Lemma A.2. \square

We use Lemma A.3 to deduce a positivity statement characterizing those weights of $\tilde{\rho}_{\mathbf{C}}$ which are weights of $\tilde{V}^{-1,0}$.

Lemma A.4. *Let ξ be a $\tilde{\mathbf{T}}_{\mathbf{C}}$ -weight of $\tilde{\rho}_{\mathbf{C}}$.*

- (1) *If ξ is a weight of $\tilde{V}^{-1,0}$, then $\langle \xi, \alpha_i^\vee \rangle \geq 0$ for all i .*
- (2) *As a partial converse, if $\langle \xi, \alpha_i^\vee \rangle > 0$ for some i , then ξ is a weight of $\tilde{V}^{-1,0}$.*

Proof. Since $\tilde{\rho}_{\mathbf{C}}$ is self-dual, its $\tilde{\mathbf{T}}_{\mathbf{C}}$ -weights are closed under $x \mapsto -x$. It follows that Part (1) and Part (2) of the lemma are equivalent. So assume ξ is a weight of $\tilde{V}^{-1,0}$ and consider Part (1).

Let ρ' be an irreducible factor of $\tilde{\rho}_{\mathbf{C}}$, which admits ξ as a $\tilde{\mathbf{T}}_{\mathbf{C}}$ -weight. Let ξ_h be the highest weight of ρ' . Since the highest weight is Δ^\vee -dominant, one has $\langle \xi_h, \alpha_i^\vee \rangle \geq 0$. We need to use the hypothesis that ξ is a weight of $\tilde{V}^{-1,0}$ to conclude that also $\langle \xi, \alpha_i^\vee \rangle \geq 0$.

Write

$$(A.3) \quad \xi_h - \xi = \sum_{\alpha \in \Delta} n(\alpha) \alpha,$$

with $n(\alpha) \geq 0$ for all $\alpha \in \Delta$.

Since μ is minuscule and α_i is noncompact, we know that $\langle \alpha_i, \mu \rangle = 1$. Since $\mu = \tilde{\mu}\nu$ with $\nu : \mathbf{G}_m \rightarrow \mathbf{G}_{\mathbf{C}}$ fractional and central, we conclude that the adjoint actions of $\mu(z)$ and $\tilde{\mu}(z)$ coincide. Hence also $\langle \alpha_i, \tilde{\mu} \rangle = 1$.

Combining our assumption that ξ is a weight of $\tilde{V}^{-1,0}$, Lemma A.2 and Lemma A.3, we have $\langle \xi_h - \xi, \tilde{\mu} \rangle = 0$. Therefore the multiplicity $n(\alpha_i) = 0$ in (A.3). A simple property of root systems says that if $\langle \alpha, \beta^\vee \rangle > 0$ for some $\alpha, \beta \in \Delta$, then $\alpha = \beta$ [Kna96, Lemma 2.51]. Hence $\langle \xi_h - \xi, \alpha_i^\vee \rangle \leq 0$. But $\langle \xi_h, \alpha_i^\vee \rangle \geq 0$ because ξ_h is Δ^\vee -dominant. So $\langle \xi, \alpha_i^\vee \rangle \geq 0$, as was to be shown. \square

The character η_ω is a character of the Levi \mathbf{L} . Therefore $\langle \eta_\omega, \alpha^\vee \rangle = 0$ for all compact roots α . Let $\tilde{\eta}_\omega$ be the restriction of η_ω to $\tilde{\mathbf{T}}_{\mathbf{C}}$. Then $\tilde{\eta}_\omega$ decomposes as a sum of components, each a character of \tilde{L}_i . Let $\eta_{\omega,i}$ be the resulting character. Suppose $\tilde{L}_i \neq \tilde{G}_i$. Then $\eta_{\omega,i}$ is a multiple, say m_i , of the fundamental weight of \tilde{G}_i dual to α_i^\vee (relative to \langle, \rangle). Recall that α_i is necessarily *special* i.e. it appears with multiplicity one in the decomposition of the highest root as a positive linear combination of simple roots [Del77, 1.2.5]. By definition, we have $m_i = \langle \eta_{\omega,i}, \alpha_i^\vee \rangle = \langle \eta_\omega, \alpha_i^\vee \rangle$.

Lemma A.5. *The multiplicities m_i are independent of i .*

Proof. Consider two pairs (\tilde{G}_i, α_i) and (\tilde{G}_j, α_j) defining the multiplicities m_i and m_j . Let $\sigma \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ conjugate \tilde{G}_i to \tilde{G}_j . Observe that the coroots $\sigma\alpha_i^\vee$ and α_j^\vee are in the same Weyl group orbit. Indeed, α_i^\vee and α_j^\vee are both special, hence have the same length. Moreover, in each of the cases A, B, C, D, two roots are in the

same Weyl group orbit if and only if they have the same length. Write $w\sigma\alpha_i^\vee = \alpha_j^\vee$ with $w \in W$.

Denote the set of $\tilde{\mathbf{T}}_{\mathbf{C}}$ -weights of $\tilde{V}^{-1,0}$ by \mathcal{S} . Given $\xi \in \mathcal{S} \cup -\mathcal{S}$, let $m(\xi)$ denote the multiplicity of ξ as a weight of $\tilde{\rho}_{\mathbf{C}}$. Then $m(\xi) = m(w\sigma\xi)$ and we have

$$m_i = \sum_{\xi \in \mathcal{S}} m(\xi) \langle \xi, \alpha_i^\vee \rangle = \sum_{\xi \in \mathcal{S}} m(\xi) \langle w\sigma\xi, w\sigma\alpha_i^\vee \rangle = \sum_{\xi \in \mathcal{S}} m(\xi) \langle \xi, \alpha_j^\vee \rangle = m_j.$$

Here the penultimate equality is a consequence Lemma A.4, because that lemma shows that all the terms in the above sums are nonnegative and $w\sigma$ maps the strictly positive terms on the left bijectively onto the strictly positive terms on the right. □

Proof of Theorem 2.1.3, completed: Since \mathbf{G} and $\tilde{\mathbf{G}}$ have the same adjoint group, one has $\langle \eta_\omega, \alpha^\vee \rangle = \langle \tilde{\eta}_\omega, \alpha^\vee \rangle$ for all roots α . It is therefore equivalent to show that $\tilde{\eta}_\omega$ is quasi-constant. Suppose a root α and $\sigma \in W \rtimes \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ satisfy $\langle \tilde{\eta}_\omega, \alpha^\vee \rangle \neq 0$ and $\langle \tilde{\eta}_\omega, \sigma\alpha^\vee \rangle \neq 0$. Let \tilde{G}_i (resp. \tilde{G}_j) be the unique factor of $\tilde{\mathbf{G}}_{\mathbf{C}}$ of which α (resp. $\sigma\alpha$) is a root. Then $\langle \tilde{\eta}_\omega, \alpha^\vee \rangle = \langle \tilde{\eta}_{\omega,i}, \alpha^\vee \rangle$ and $\langle \tilde{\eta}_\omega, \sigma\alpha^\vee \rangle = \langle \tilde{\eta}_{\omega,j}, \sigma\alpha^\vee \rangle$. By Lemma A.5, one has $\tilde{\eta}_{\omega,i} = mf_i$ (resp. $\tilde{\eta}_{\omega,j} = mf_j$), where $m = m_i = m_j$ and f_i (resp. f_j) is the fundamental weight dual to α_i^\vee (resp. α_j^\vee).

In cases *A* and *D*, the fundamental weights f_i and f_j are minuscule, hence $|\langle f_i, \alpha^\vee \rangle| = |\langle f_j, \sigma\alpha^\vee \rangle| = 1$ (by the assumptions above both pairings are nonzero).

In cases *B* and *C*, the Weyl group has two orbits on the set of roots (resp. coroots), consisting of the long roots (resp. coroots) and the short roots (resp. coroots). The pairing $\langle f_i, \alpha^\vee \rangle$ has value 1 if α^\vee is short and 2 if α^\vee is long (again because the pairing was assumed nonzero). We conclude by observing that the property of being long (resp. short) is preserved under $W \times \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. □

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